JOHAN'S PROBLEM SEMINAR

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I'm writing this as a practice of my latex and my english (maybe also my math I guess?). So please tell me if you see any mistake of these kinds...

1. Remy's question

Question 1.1. *Let $f : X \to Y$ be an étale morphism of varieties over an algebraically closed field. Assume that all fibres have the same size. Is f necessarily finite étale?*

Theorem 1.2. *We prove a more general version: just assume $f : X \to Y$ is a separated morphism between two locally Noetherian schemes.*

Proof. So now one only have to prove such a map is proper or, equivalently, finite. To prove properness, we may use valuative criterion. So we can reduce to the case where Y is the spectrum of a DVR. The claim is that in this case such a morphism must be finite. As finiteness is permanent (and also descent) after a fppf base change, we can base change to strict henselization of Y. And in this case X is automatically gonna be an open subset of finite copy of Y(here we used the morphism is separated). As the constancy of fibers is preserved by base change, we see immediately X IS actually a finite copy of Y in this case. □

Now I want to illustrate the importance of assuming f to be separated by giving following example which Johan told me last Friday.

Example 1.3. *Let X be disjoint union of an affine line minus origin and another affine line with double origins, Y be an affine line. The natural map from X to Y is étale and everywhere same number of fibers yet not finite étale. The example above is, of course, not separated. This phenomenon corresponding to the criterion of a constructible sheaf being locally constant (specialization map to be bijective, which is not the case in this example).*

2. Dave's question

Let $X = (|X|, O_X)$ be a ringed space with an action of a finite group G. Then we can form the ringed space $X/G = (|X|/G, (O_X)^G)$; this thing is initial for G-equivariant maps from X to ringed spaces w/ trivial G-actions.

Question 2.1.

- *Problem A: Give an example where X is locally ringed but X/G is not *locally* ringed.*
- *Problem B: Give an example where X is a scheme but X/G is not a scheme.*

(There is some obvious overlap here.) Note that with regard to problem B, it's proved in SGA 1 (Prop. V.1.8) that if X is a scheme and each G-orbit of points is contained in an open affine subscheme, then X/G is a scheme.

I can prove the example in Problem A doesn't exist. And Hironaka had an example of Problem B, see 'Hironaka's example' in Wikipedia.

Proposition 2.2. *Let $X = (|X|, O_X)$ be a locally ringed space with an action of a finite group G. Then X/G is also locally ringed.*

We can first see an example:

Example 2.3. *Let X be the affine line with double origins. Consider following $G = \mathbb{F}_2$ action on X: σ maps $\mathbb{G}_m$ to $\mathbb{G}_m$ by sending x to $-x$. Now extend σ to an automorphism on X by interchanging the two origins. The quotient is actually a $\mathbb{A}^1_k$! Let's compute carefully the stalk of $O_{X/G}$ at the origin. It's gonna be $\lim_{x \in U}\{f \in O_X(U) : f|_{U\cap\sigma U} = \sigma f|_{U\cap\sigma U}\}$, where x is one of the origins. Which is NOT the same as $(O_{X,x} \times O_{X,\sigma x})^G$ (did you see this, Dave? :).*

Okay, after meditating about this example for a while, we are ready to prove the proposition above.

Proof. We just observe that

$$O_{X/G,\bar{x}} = \lim_{x\in U}\{f \in O_X(U) : f|_{U\cap\sigma U} = \sigma f|_{U\cap\sigma U}\} \hookrightarrow O_{X,x}.$$

where $\bar{x}$ is the image of x under this quotient map and $\sigma f(\sigma x) := f(x)$. Here we used that G is a finite group.

Claim The intersection $m_x \cap O_{X/G,\bar{x}}$ is the only maximal ideal of $O_{X/G,\bar{x}}$, where m_x is the maximal ideal of $O_{X,x}$.

Proof of the claim: we only have to prove every $f \in O_{X/G,\bar{x}} - m_x$ is invertible in $O_{X/G,\bar{x}}$. But now that f is not in m_x, it's invertible in $O_{X,x}$ with inverse, say, f^{-1}. We only have to check f^{-1} lies in $O_{X/G,\bar{x}}$, i.e., it satisfies the condition above. We may assume both f and f^{-1} lies in $O_X(U)$, then

$$1 = f|_{U\cap\sigma U} \times f^{-1}|_{U\cap\sigma U} = \sigma f|_{U\cap\sigma U} \times \sigma f^{-1}|_{U\cap\sigma U}$$

As f is σ invariant, we see that

$$f|_{U\cap\sigma U} \times (f^{-1}|_{U\cap\sigma U} - \sigma f^{-1}|_{U\cap\sigma U}) = 0$$

Now f is invertible on U, hence invertible on $U \cap \sigma U$, therefore we have:

$$f^{-1}|_{U\cap\sigma U} = \sigma f^{-1}|_{U\cap\sigma U}$$

which is what we want to prove. □

There is the following proposition which is somewhat related to what we have discussed above:

Proposition 2.4. *Let (R, m) be a local ring and suppose $\sigma_i : R \to R$ are automorphisms of R. Then $R^{\{\sigma_i\}}$ is a local ring.*

Proof. One only has to prove if $r \in R$ is invertible and σ invariant, then the inverse is also gonna be σ invariant. Let's say $rr' = 1$, then $\sigma r \sigma r' = 1$, so $r \times (r' - \sigma r') = 0$ because r is σ invariant. Therefore we see that $r' = \sigma r'$, which is what we wanted to prove.

□

3. Qixiao's question

To keep me safe, let's work over a field this time.

Question 3.1. *How to prove the map $\pi : GL_n \to PGL_n$ has no section?*

Let's prove a more general result, observe that determinant is an irreducible homogeneous polynomial of degree n.

Proposition 3.2. *Let F be a homogeneous irreducible degree d polynomial in $n+1$ variables. The map $\pi : D(F) \to D_+(F)$ has a section if and only if d=1.*

We are following notations in Hartshorne's Algebraic Geometry. $D(F) = \mathrm{Spec}k[x_1, ..., x_{n+1}, F^{-1}]$, $D_+(F) = \mathrm{Spec}k[x_1, ..., x_{n+1}, F^{-1}]_0$, degree 0 part of the ring of $D(F)$.

Proof. Obviously π will have a section if F is linear. On the other hand, suppose π has a section. This is amounts to saying that we have $n+1$ functions $\frac{f_i}{F^k}, i = 0, ..., n$ such that $[f_0, ..., f_n] = [x_0, ..., x_n]$, i.e., $f_i = x_i g$ without common zero: $\exists g_i$ such that $\sum f_i g_i = F^{k+l}$. Now as $f_i = x_i g$, we see that $g \times \sum x_i g_i = F^{k+l}$. Since F is irreducible, g will be a power of F, hence $1 + k_1 \mathrm{deg} F = k \mathrm{deg} F$, so F is linear. □

Daniel pointed out a geometric way to settle down the question of what will happen if F is not irreducible:

Proposition 3.3. *Let F be a homogeneous degree d polynomial in $n+1$ variables. The map $\pi : D(F) \to D_+(F)$ has a section if and only if gcd of degree of prime factors of F equals 1.*

Proof. One may regard $\mathbb{A}^{n+1} \setminus \{0\}$ as total space of $O(-1)$ deleting zero sections over $\mathbb{P}^n$. And find a section after restricting to $D_+(F)$ is the same as trivialize $O(-1)|_{D_+(F)}$. It's well known that Picard group of $D_+(F)$ is just $\mathbb{Z}/d$ with generator $O(1)$, where d is gcd of degree of prime factors of F. □

4. Daniel's Question

Question 4.1. *Given a diagonalizable matrix $A \in GL_n(\mathbb{Z}_p)$ with distinct eigenvalues, is it true that any small perturbation $A + \epsilon B$ will also be diagonalizable in $GL_n(\mathbb{Z}_p)$?*

Let me elaborate more on the question. Begin diagonalizable means there is an matrix $P \in GL_n(\mathbb{Z}_p)$ such that PAP^{-1} is diagonal. Then the eigenvalues of A will automatically be in $mathbbZ_p$, so does any small perturbation of A.

I don't know how to do it in general, mainly due to my lack of knowledge about finite $mathbbZ_p[x]$. What I know how to prove is the following

Proposition 4.2. *Let A be as above, with an extra condition that its distinct eigenvalues remains distinct after modulo p (admittedly this condition makes this proposition uninteresting), then for any small perturbation $A + \epsilon B$ will also be diagonalizable in $GL_n(\mathbb{Z}_p)$.*

Proof. Let V be a finite $\mathbb{Z}_p$ module on which A acts. Then by my extra condition we can find an element $v \in V$ such that the map $\mathbb{Z}_p[x] \to V$ sending $P(x)$ to $P(A)v$ is surjective. Now for any small perturbation, the same $v \in V$ will also exhibit a surjective map. Hence making V a quotient of $\mathbb{Z}_p[x]$, now we only have to think about the kernel of this quotient. Apparently the eigenpolynomial $det(xI-(A+\epsilon B))$ will lie in this ideal, and after tensoring with $\mathbb{Q}_p$ they are the same. Now by some flatness argument we see that $V = \mathbb{Z}_p[x]/(P)$ with respect to the action $A+\epsilon B$ on the left and multiplication by x on the right, where P is a polynomial whose roots lies in $\mathbb{Z}_p$. Such a module can be diagonalized.

□

One may wondering if a matrix diagonalizable in $GL_n(mathbbQ_p)$ will automatically going to be diagonalizable in $GL_n(mathbbZ_p)$, however this is not the case. One can take a lift of a non-diagonalizable matrix in $GL_n(\mathbb{F}_p)$ with distinct eigenvalues, which will be a counterexample of our confusion above.

5. Dingxin's Question

This question is about real AG, really fun.

Question 5.1. *We all know that on a smooth cubic surface over algebraically closed field, there will be 27 lines. Now if the surface is defined over $\mathbb{R}$, what are the possible numbers of real lines?*

Let's recall that every smooth cubic surface is a blow-up of $\mathbb{P}^2$ at 6 points. Which 6 points will make the blow-up naturally be defined over $\mathbb{R}$? A natural answer is that the points are consisted of n pairs of conjugate complex points and $6-2n$ real points. There are 4 cases, let's compute them carefully.

In view of blow-up of $\mathbb{P}^2$ at 6 points, the 27 lines are divided into 3 classes: (total transformation of) lines connecting 2 points (15 of them), conics passing thru 5 points (6 of them) and 6 exceptional divisors. In the situation above, there will be $\binom{6-2n}{2}+n$ real lines, $6-2n$ real conics and $6-2n$ real exceptional divisors. If one plugs $n = 0, 1, 2, 3$ in the formulae above, one would see that at least the number can be 3, 7, 15 and 27. So the remaining question is do we run out of all possibilities?

One can actually prove the following Proposition:

Proposition 5.2. *On a cubic surface defined over $\mathbb{R}$, one can find a set of 6 lines don't intersecting each other and the set is preserved under conjugation of $\mathbb{C}$ over $\mathbb{R}$.*

The proof uses a fact that the automorphism group of these 27 lines is the Weyl group of E_6, and do some combinatorics. Admitting this proposition, one would recognize that any real cubic surface arises as above. So yeah, we run out of all possibilities.

6. Dave's Question

Question 6.1. *Suppose A is a ring over $\mathbb{F}_p$ such that $\Omega_{A/\mathbb{F}_p} = 0$, is Frobenius surjective on A?*

Dan's answer is no, and I'll try to explain his construction in this section. But before that let's agree that counterexample of above must be a disgustingly huge ring by proving the following:

Proposition 6.2. *With the above assumption, let's assume furthermore that A is finitely generated over $\mathbb{F}_p$, then Frobenius is surjective on A.*

Proof. A is formally étale over $\mathbb{F}_p$, being finitely generated over it makes A étale over it. Hence A will be nothing but product of finitely many finite extension of $\mathbb{F}_p$. Therefore Frobenius is surjective on A. □

So here comes Dan's idea. Given a graded $\mathbb{F}_p$-algebra A, and two (possibly same) homogeneous elements a and b. He considered

$$A' = A[x_1, y_1, \cdots, x_n, y_n, z]/(a - x_1^2 y_1^p, x_1 y_1^p - x_2^2 y_2^p, \cdots, x_{n-1} y_{n-1}^p - x_n^2 y_n^p, x_n y_n^p - z^p)$$

. And he made the following:

Claim 6.3. *$da = 0$ for whatever n is. And with n big enough, the 'new' p-th power in A will have big degree (so doesn't contain b).*

Proof. Firstly let's agree that $da = 0$ in $\Omega_{A'/\mathbb{F}_p}$. Indeed, we have

$$da = 2x_1 y_1^p dx_1, y_1^p dx_1 = 2x_2 y_2^p dx_2, \cdots, y_n^p dx_n = 0$$

Secondly let's agree that $(A')^p \cap A$ is generated by A^p and powers of a, where A^p is just the subring of p-th power elements in A. Let's choose a set of Gröbner bases: $\{x^\alpha y^\beta z^k; k < p, \alpha_i \beta_i = (m, l)$ such that $l \in \mathbb{N}$ if $m = 0$ or 1, and $l < p$ otherwise$\}$. So suppose an element $a' = \sum r_{\alpha\beta k} x^\alpha y^\beta z^k$ has p-th power in A. We see that $a'^p = \sum r^p_{\alpha\beta k} x^{p\alpha} y^{p\beta} z^{pk} = \sum s^p_{\alpha\beta kl} x^\alpha y^\beta z^k a^l$, so it's in A if and only if $a' = \sum s_l^p a^l$ as we chose Gröbner bases in such a way. Hence follows at the beginning of this paragraph.

Last thing to check is that when n gets bigger and bigger, the smallest power of a in $A'^p \cap A - A^p$ gets bigger and bigger. It's easy to see (by choosing weights on x_i, y_i, z in such a way that weights of x_i, y_i are the same) that the powers of a in $A'^p \cap A - A^p$ always come as a p-th power of monomials in $A[x_1, y_1, \cdots, x_n, y_n, z]$. So now suppose $p \not| \alpha_0$

$$a^{\alpha_0} = x_1^{2\alpha_0} y_1^{p\alpha_0} = x_1^{2\alpha_0 - \alpha_1} y_1^{p(\alpha_0 - \alpha_1)} x_2^{2\alpha_1} y_1^{p\alpha_1} = \cdots =$$

$$= x_1^{2\alpha_0 - \alpha_1} y_1^{p(\alpha_0 - \alpha_1)} x_2^{2\alpha_1 - \alpha_2} y_2^{p(\alpha_1 - \alpha_2)} \cdots x_n^{2\alpha_{n-1} - \alpha_n} y_n^{p(\alpha_{n-1} - \alpha_n)} z^{p\alpha_n}$$

We see immediately that for such a thing being able to take p-th roots in A', the condition on α_i's are $\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_0$, $p \not| \alpha_i$ and $p | (2\alpha_{i-1} - \alpha_i)$. Therefore we have $\alpha_{i-1} \geq \alpha_i + 1$, leaving us $\alpha_0 \geq n$. Hence the new p-th powers in A' will have degree at least $n \cdot \deg(a)$. □

Now let's start with $A = \mathbb{F}_p[x]$, and grade it such that x has degree 1. And we keep applying the above construction, where we choose a to be any homogeneous element with positive degree and $da \neq 0$ and fix $b = x$. Notice that degree 0 part never changes in our procedure and it's always gonna be $\mathbb{F}_p$. After a transfinite induction, we reach to a ring A', where there's no Kähler differential yet x has no p-th root.

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