WHY ARE OPEN/CLOSE DISCS, $\mathbb{A}^1$ AND $\mathbb{P}^1$ NOT SIMPLY CONNECTED?

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1. Introduction and examples for discs

This is the notes for a talk given in Johan's seminar on examples. The paper I choose to talk is [dJ95].

Throughout this talk, let X be a quasi-separated quasi-compact rigid space over a non-Archimedean field.

According to Grothendieck, fundamental group of a space is defined via declaring covering spaces. Any reasonable definition of "covering spaces" of a rigid space X should include its finite étale covers as examples. In particular, if a rigid space X has nontrivial finite étale cover then it is not simply connected. Let us first justify our title by giving finite étale covers of open/close disc of radius 1 over $\mathbb{C}_p$.

Example 1.1. Let $\mathbb{D}$ denote the open disc of radius 1 over $\mathbb{C}_p$. For each n we have a finite étale cover of $\mathbb{D}$ by itself which sends x to $(1+x)^{p^n}-1$. Viewing $\mathbb{D}$ as the formal fibre of the formal group scheme $\widehat{\mathbb{G}_m}$, our cover is the restriction of raising to p^n-th power of $\widehat{\mathbb{G}_m}$.

These covers give rise to a surjection $\pi_1(\mathbb{D}) \twoheadrightarrow \mathbb{Z}_p^{\times}$.

Example 1.2. Let $\mathbb{B}$ denote the close disc of radius 1 over $\mathbb{C}_p$. One way to find some finite étale cover, we can try to "deform" the Artin–Schreier map in characteristic p. Namely we have the map from $\mathbb{B}$ to itself which sends x to $x^p - x$. So we conclude that $\mathbb{B}$ is not simply connected.

2. Definitions and example for $\mathbb{A}^1$

The following Definition is copied from [dJ95, Definition 2.1]

Definition 2.1. A morphism $f\colon X \to Y$ of rigid spaces is called an *étale covering map* (resp. *topological covering map*) if there exists a (<u>wide</u> admissible) open covering $\{U_i\}$ of Y such that

$$X \times_Y U_i = \coprod V_{ij} \text{ where each } V_{ij} \to U_i \text{ is finite étale (resp. isomorphism).}$$

In the situations above we call Y an *étale covering space* (resp. *topological covering space*) of X.

Here wide opens are the same as opens come from the corresponding Berkovich space. Another way to think of them is that they are essentially defined by inequalities (instead of equalities) of norms. If you are an adic person, then these are the open subsets which are stable under specialization. Compare with [FvdP04, Exercise 7.1.12].

Now given a geometric point $\bar{x} \to X$, for any $M \xrightarrow{\pi} X$ an étale covering space (resp. finite étale covering space, resp. topological covering space), we can get a set $\{\pi^{-1}(\bar{x})\}$. This defines a fiber functor

$$F_{\bar{x}} \colon \begin{Bmatrix} \text{étale covering space (resp. topological covering} \\ \text{space, resp. finite étale covering space) of } X \end{Bmatrix} \to \mathcal{S}\text{ets}.$$

One can verify that these three categories, when equipping the fiber functor above, are Galois categories. For a crash course on Galois cats, see [Lee15, Lectures 2 & 3]. Consequently, we may make the following

Definition 2.2. The groups of automorphisms of $F_{\bar{x}}$ on the three Galois categories above are called the *fundamental group* (resp. *topological fundamental group*, resp. *algebraic fundamental group*) of X and are denoted by $\pi_1(X, \bar{x})$ (resp. $\pi_1^{\mathrm{top}}(X, \bar{x})$, resp. $\pi_1^{\mathrm{alg}}(X, \bar{x})$). These are viewed as topological groups with topology coming from stabilizers of $\bar{y} \in F_{\bar{x}}(Y)$ where $Y \to X$ is a covering space of the corresponding type.

The first question one could ask is whether the definitions above depend on our choice of a geometric point $\bar{x}$. Fortunately, we have the following theorem (copied from [dJ95, Theorem 2.9]).

Theorem 2.3 (dJ)**.** *Suppose X is connected. For any two geometric points $\bar{x}, \bar{x}'$ there exists an isomorphism of functors $F_{\bar{x}} \cong F_{\bar{x}'}$.*

For rigid analytification of algebraic curves, one can actually compute the topological and algebraic fundamental groups. In particular, we have the following

Proposition 2.4. *Let $X = \mathbb{A}^1_{\mathbb{C}_p}$ or $\mathbb{P}^1_{\mathbb{C}_p}$, then we have*

$$\pi_1^{\mathrm{top}}(X, \bar{x}) = \pi_1^{\mathrm{alg}}(X, \bar{x}) = 0.$$

Proof. The computation of π_1^{top} is due to Berkovich. [Ber90, Theorem 6.1.5] says that any connected open subspace of $\mathbb{P}^{1,\mathrm{rig}}$ is topologically simply connected.

As for the computation of π_1^{alg}, one may use GAGA and the comparison of algebraic fundamental group and the fundamental group as in topology. □

Although both of rigid affine line and rigid projective line do not have topological/algebraic covering. We will see that they still have gigantic fundamental groups.

Example 2.5. Consider

$$\begin{aligned} \log \colon \mathbb{D} &\to \mathbb{A}^1_{\mathbb{C}_p}, \\ 1 - x &\mapsto x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

This is an étale covering space. This can be seen by noticing:

- for a small (open) disc U around $0 \in \mathbb{A}^1$, we can define exponential,
- $\{\frac{1}{p^n}U\}_n$ gives a wide admissible open covering of $\mathbb{A}^1$ and,
- we have a commutative diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\log} & \mathbb{A}^1 \\ \downarrow{\scriptstyle (\cdot)^{p^n}} & & \downarrow{\scriptstyle p^n} \\ \mathbb{D} & \xrightarrow{\log} & \mathbb{A}^1 \end{array}$$

When base change to $\mathbb{C}_p$ we get a Galois covering with Galois group $\mu_\infty(\mathbb{C}_p)$. Therefore we have a continuous quotient

$$\pi_1(\mathbb{A}^1_{\mathbb{C}_p}) \twoheadrightarrow \mu_\infty(\mathbb{C}_p)$$

where the later group is given discrete topology.

So as a matter of fact, in our p-adic geometry world, $\mathbb{A}^1_{\mathbb{C}_p}$ is NOT simply connected.

3. Period map and example for $\mathbb{P}^1$

Let $E/\mathbb{F}_p$ be a supersingular elliptic curve. The existence of such a thing follows, by Honda–Tate theory, merely from the fact that one can write down a polynomial $x^2 + p$.

The universal deformation ring (á la Schlessinger) of E is $\mathbb{Z}_p[\![T]\!]$. This just means that we have

$$\begin{array}{c} \mathcal{E} \\ f \downarrow \;\; \uparrow s \\ \mathrm{Spf}(\mathbb{Z}_p[\![T]\!]) \end{array}$$

aka, the pro-representing family of elliptic curves. Take their associated "generic fiber" we get

$$\begin{array}{c} \mathcal{E}^{\mathrm{rig}} \\ f \downarrow \;\; \uparrow s \\ \mathbb{D} \end{array}$$

a family of elliptic curves over open disc which can be thought of as "the universal family of elliptic curves with given reduction E". Although we have declared $\mathbb{D}$ to be the open disc over $\mathbb{C}_p$, now we only view it as an open disc over $\mathbb{Q}_p$. The whole picture, as for now, still lives over $\mathbb{Q}_p$. However from now on, we will base change to $K_0 := W(\bar{\mathbb{F}_p})[\frac{1}{p}]$ and denoted everything with the same symbol. Since we have not heard any objection (as there is really no audience for now), we will still denote $\mathcal{E}^{\mathrm{rig}}$ by $\mathcal{E}$. By (derived)-pushing-forward the relative (analytic/continuous) differential complex, we get

$$\mathbb{R}^1 f_* \Omega^\bullet_{\mathcal{E}/\mathbb{D}} \xrightarrow{\nabla} \mathbb{R}^1 f_* \Omega^\bullet_{\mathcal{E}/\mathbb{D}} \otimes_{\mathcal{O}_\mathbb{D}} \Omega^1_\mathbb{D}.$$

Let us denote $\mathbb{R}^1 f_* \Omega^\bullet_{\mathcal{E}/\mathbb{D}}$ by $\mathcal{V}$, it is a vector bundle equipped with an integrable connection. It is the analogous story as in complex algebraic geometry, and one should also view this (Gauss–Manin) connection as a system of (p-adic) ODEs. The first input we need is the following Theorem/Fact which is often called Dwork's trick.

Fact 3.1 (Dwork's trick)**.**

$$\mathcal{V} = \ker(\nabla) \otimes_{K_0} \mathcal{O}_\mathbb{D}.$$

This just means that our system of ODEs can be solved on the whole open disc. Dwork's trick was that first solving it on a small disc around 0 and then use Frobenius structure (coming from that on $E_{\bar{\mathbb{F}_p}}$) to argue that these solutions can be extended to a bigger disc.

Recall that we have a short exact sequence from the spectral sequence:

$$0 \to f_* \Omega^1_{\mathcal{E}/\mathbb{D}} \to \mathcal{V} \to R^1 f_* \mathcal{O}_{\mathcal{E}/\mathbb{D}} \to 0.$$

Similar to the period map defined in classical complex geometry, next thing we need is a global section of $f_*\Omega^1_{\mathcal{E}/\mathbb{D}}$. We just notice that $[s(\mathbb{D})]$ is a relatively ample divisor, and analogously we conclude that $\mathcal{E}$ can be presented as $\{y^2 = x^3 + a_1(T)x + a_0(T)\}$ where $a_i(T)$ are analytic functions on $\mathbb{D}$. Now $\omega = \frac{dx}{y} \in f_*\Omega^1_{\mathcal{E}/\mathbb{D}}$ is the global 1-form we have been sought for. This ω gives rise to a map

$$\pi \colon \mathbb{D} \to \mathbb{P}^1(\ker(\nabla))$$

which we call the period map. This is a map defined over K_0. In concrete terms, we may choose a K_0-basis of $\ker(\nabla)$, say, $\langle e, f\rangle$. Then by Fact 3.1, we may write $\omega = a(T)e + b(T)f$. So with respect to coordinates corresponding to this chosen basis, $\pi(T) = [a(T) \colon b(T)]$.

We claim that this is an étale covering space. Let us first see why this is an étale map.

Proposition 3.2. *π is étale.*

Proof. This is because $d\pi$ is the Kodaira–Spencer map which is an isomorphism by our setup (since $\mathbb{D}$ is "the moduli"). □

In particular, π is an open map. Hence its image is an open subspace

$$\mathrm{Im}(\pi) =: \mathbb{P}^{1,\mathrm{adm}} \subset \mathbb{P}^1.$$

On the other hand we have, $\mathbb{L} = R^1 f_{\acute{e}t,*}(\mathbb{Q}_p)_{\mathcal{E}}$, a rank 2 local system on $\mathbb{D}$. We need the following

Fact 3.3. *There exists a $\mathbb{Q}_p$ local system $\mathbb{M}$ on $\mathbb{P}^{1,\mathrm{adm}}$ and a natural isomorphism $\mathbb{L} \cong \pi^*\mathbb{M}$.*

Proof. Let us say a bit of heuristic here. Recall that Fontaine's functor $\mathbb{D}_{crys}$ associates any p-adic representation with an (admissible) filtered F-isocrystal and it actually defines an equivalence of (abelian!) categories (with adjective "Crystalline" on the p-adic representation side). One can make this statement into a relative version, namely, one can associate any $\mathbb{Q}_p$ local system with a family of (admissible) filtered F-isocrystals. After defining what a $\mathbb{Q}_p$ local system being Crystalline means, one can prove that this association gives a fully faithful embedding of Crystalline local systems to family of admissible filtered F-isocrystals.

In our situation, the association $\mathbb{L} \leftrightsquigarrow (\ker(\nabla), \omega)$ is an example of the aforesaid relative version. Now by fully faithfulness and the fact that $(\ker(\nabla), \omega)$ is a family of filtered constant F-isocrystals pulled back from $\mathbb{P}^{1,\mathrm{adm}}$, we see that one has a natural descent datum for $\mathbb{L}$ which gives rise to our $\mathbb{M}$. □

Now let $\mathcal{M}$ be the "space parametrizing lattices in $\mathbb{M}$" as defined in [dJ95, first paragraph of proof of Theorem 4.2]. In loc. cit. it is shown that any space arising as above would be an étale covering space. Let us recall another

Fact 3.4. *We have decomposition $\mathcal{M} = \coprod_n \mathcal{M}^{(n)}$ and a natural isomorphism $\mathbb{D} \cong \mathcal{M}^{(0)}$ over $\mathbb{P}^1$.*

Proof. Let us say a few heuristic on this. Inside $\mathbb{L}$ (which (étale)-locally is the same as $\mathbb{M}$) we have a natural choice of lattices, namely the family of integral Tate module of $\mathcal{E}$. Therefore any other $\mathbb{Z}_p$-lattice can be compared with these ones, and the degree is a discrete invariant. This is why our space $\mathcal{M}$ decomposed into pieces indexed by integers.

As for $\mathbb{D} \cong \mathcal{M}^{(0)}$, Serre–Tate theory tells us that deforming an elliptic curve (with residue characteristic p) is the same as deforming its p-divisible group or equivalent its Tate module. This is also analogous to the classical complex geometry where we know that (fixing complex plane) deforming elliptic curve is the same as deforming lattice in complex plane. □

Now we can draw a partial conclusion as below.

Corollary 3.5. $\mathbb{D} \to \mathbb{P}^{1,\mathrm{adm}}$ *is an étale covering space.*

Therefore it suffices to show that π is a surjection.

Theorem 3.6 (Gross–Hopkins)**.** $\mathbb{P}^{1.\mathrm{adm}} = \mathbb{P}^1$.

Proof. In [HG94, Corollary 23.15 & Corollary 23.21], Gross and Hopkins discovered the following. If we choose a set of basis (e, f) of $\ker(\nabla)$ such that "the semi-linear Frobenius" acts as

$$\begin{bmatrix} 0 & 1 \\ p^{-1} & 0 \end{bmatrix}.$$

Then the image of $\mathbb{B}(\frac{1}{2})$ is $\left\{[1 : w] \,\middle|\, v_p(w) \geq \frac{1}{2}\right\}$. Now the map $\pi\colon \mathcal{M} \to \mathbb{P}^1$ has a equivariant action under Q^* where Q is the nontrivial quaternion algebra over $\mathbb{Q}_p$. Also the action of $\sqrt{p}$ on $\mathbb{P}^1$ (w.r.t. our chosen basis) is given by

$$\begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}.$$

One can check directly that aforesaid open subspace and its image under action of $\sqrt{p}$ forms an admissible cover of $\mathbb{P}^1$. □

Remark 3.7. One really needs to use the $\mathcal{M}$ to see surjectivity of π since our element $\sqrt{p}$ shift the components of $\mathcal{M}$ by 1 (or -1 if you choose another way to order them. . .).

Analogous to what happens in complex/algebraic geometry, $\mathbb{M}$ gives rise to a homomorphism

$$\pi_1(\mathbb{P}^1_{K_0}, \bar{x}) \to \mathrm{GL}_2(\mathbb{Q}_p).$$

To see what the image is, we first notice the following

Proposition 3.8. *The following diagram*

$$\begin{array}{ccc} \pi_1(\mathbb{P}^1_{K_0}, \bar{x}) & \longrightarrow & \mathrm{GL}_2(\mathbb{Q}_p) \\ \downarrow & & \downarrow \scriptstyle{\det} \\ \mathrm{Gal}(\mathbb{C}_p/K_0) & \xrightarrow{\chi_{cyc}} & \mathbb{Q}_p^* \end{array}$$

is commutative.

Here the nameless vertical arrow comes from the structure map, and χ_{cyc} is just the cyclotomic character.

Proof. One can see this in two ways:

(1) either because we have $\wedge^2 H^1_{ét}(E_{\bar{K}_0,\mathbb{Q}_p}) \cong \mathbb{Q}_p(-1)$ or;
(2) $\wedge^2 \mathcal{O}(\frac{1}{2}) = \mathcal{O}(1)$ where we borrowed the notations of vector bundles on FF curve to denote our constant F-isocrystals.

□

Now if we denote Γ (resp. $\Gamma_{\mathbb{C}_p}$) the image of $\pi_1(\mathbb{P}^1_{K_0}, \bar{x})$ (resp. $\pi_1(\mathbb{P}^1_{\mathbb{C}_p}, \bar{x})$) in $\mathrm{GL}_2(\mathbb{Q}_p)$. We see that

$$\Gamma \subset \mathrm{GL}_2'(\mathbb{Q}_p) = \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid v_p(\det(g)) = 0\}$$

and

$$\Gamma_{\mathbb{C}_p} \subset \mathrm{SL}_2(\mathbb{Q}_p).$$

Theorem 3.9 (dJ)**.** *The two containments discussed above are equalities.*

Before proving this Theorem, we need some preparations.

Theorem 3.10 (Margulis)**.** *The only closed non-discrete subgroup with finite covolume in* $\mathrm{SL}_2(\mathbb{Q}_p)$ *is* $\mathrm{SL}_2(\mathbb{Q}_p)$ *itself.*

This is a rather special form of [Mar91, Theorem 5.1 on P95]. Note that since SL_2 is unimodular, the condition on covolume is well-defined.

Fact 3.11. *There exists a p-divisible group over* $W(\bar{\mathbb{F}}_p)$ *with special fiber* $E[p^\infty]$ *whose corresponding Galois representation has image finite index in* $\mathrm{GL}_2(\mathbb{Z}_p)$*.*

I do not know a very enlightening explanation for why this should be true. The argument in loc. cit. goes by constructing explicit example and arguing it has the required properties by using knowledge about closed subgroups in $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof (very sketchy). Now since $\pi_1(\mathbb{P}^1_{K_0}, \bar{x}) \twoheadrightarrow \mathrm{Gal}(\mathbb{C}_p/K_0) \twoheadrightarrow \mathbb{Z}_p^\times$. To show the equality for Γ, it suffices to show that $\Gamma \cap \mathrm{SL}_2(\mathbb{Q}_p) = \mathrm{SL}_2(\mathbb{Q}_p)$.

Fact 3.11 tells us that there exists a point $x \in \mathbb{D}(K_0)$ s.t. $\pi_1(x, \bar{x}) \to \pi_1(\mathbb{D}, \bar{x}) \to \pi_1(\mathbb{P}^1_{K_0}, \bar{x}) \to \mathrm{GL}_2(\mathbb{Q}_p)$ has image finite index in $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence Γ is an open subgroup in $\mathrm{GL}_2(\mathbb{Q}_p)$. In particular, $\Gamma \cap \mathrm{SL}_2(\mathbb{Q}_p)$ is a closed non-discrete subgroup in $\mathrm{SL}_2(\mathbb{Q}_p)$. Therefore, by Theorem 3.10, it suffices to show it has finite covolume.

We shall be staring at the map of double cosets

$$\Gamma \cap \mathrm{SL}_2(\mathbb{Q}_p) \backslash \mathrm{SL}_2(\mathbb{Q}_p)/\mathrm{SL}_2(\mathbb{Z}_p) \to \Gamma \backslash \mathrm{GL}_2(\mathbb{Q}_p)/p^{\mathbb{Z}} \cdot \mathrm{GL}_2(\mathbb{Z}_p).$$

We can say the following three things about it:

(1) $\Gamma \backslash \mathrm{GL}_2(\mathbb{Q}_p)/p^{\mathbb{Z}} \cdot \mathrm{GL}_2(\mathbb{Z}_p)$ is a set of two elements. This is because without quotienting out $p^{\mathbb{Z}}$ we get the set of components of $\mathcal{M}$, and p shifts the components by 2 since our lattice has rank 2.
(2) each fibre of this map can be (continuously) surjected by pro-finite spaces. Therefore $\Gamma \cap \mathrm{SL}_2(\mathbb{Q}_p) \backslash \mathrm{SL}_2(\mathbb{Q}_p)/\mathrm{SL}_2(\mathbb{Z}_p)$ has finite volume.
(3) $\mathrm{SL}_2(\mathbb{Z}_p)$ has finite volume.

Therefore we see that $\Gamma \cap \mathrm{SL}_2(\mathbb{Q}_p) \subset \mathrm{SL}_2(\mathbb{Q}_p)$ has finite covolume.

This finishes the part concerning Γ. Now we would be done if

$$\pi_1(\mathbb{P}^1_{\mathbb{C}_p}, \bar{x}) \to \pi_1(\mathbb{P}^1_{K_0}, \bar{x}) \to \mathrm{Gal}(\mathbb{C}_p/K_0) \to 0$$

is exact. But unfortunately, we only know that the image of $\pi_1(\mathbb{P}^1_{\mathbb{C}_p}, \bar{x})$ is dense in the kernel of middle arrow. Therefore, we know $\overline{\Gamma_{\mathbb{C}_p}} = \mathrm{SL}_2(\mathbb{Q}_p)$. But actually in [dJ95, P116] it was shown that $\pi_1(\mathbb{D}_{\mathbb{C}_p}, \bar{x})$ surjects onto an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$. Therefore we may conclude that $\Gamma_{\mathbb{C}_p}$ is an open group, hence is closed, hence is the whole $\mathrm{SL}_2(\mathbb{Q}_p)$. □

One might ask what happens if we start with an ordinary elliptic curve? In this case, it would actually be the log map we described before. The admissibility of a filtered F-isocrystal actually forces the image of our period map lands in $\mathbb{A}^1$. Indeed our F-isocrystal in this case decomposes into $\mathcal{O} \oplus \mathcal{O}(1)$ and our ω can never agree with $\mathcal{O}$, for otherwise the sub-filtered-F-isocrystal corresponding to $(\mathcal{O}, (\omega))$ would have Hodge polygon higher than Newton polygon which violates the admissibility.

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