ON THE u^∞-TORSION SUBMODULE OF PRISMATIC COHOMOLOGY

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Abstract. We investigate the maximal finite length submodule of the Breuil–Kisin prismatic cohomology of a smooth proper formal scheme over a p-adic ring of integers. This submodule governs pathology phenomena in integral p-adic cohomology theories. Geometric applications include a control, in low degrees and mild ramifications, of (1) the discrepancy between two naturally associated Albanese varieties in characteristic p, and (2) kernel of the specialization map in p-adic étale cohomology. As an arithmetic application, we study the boundary case of the theory due to Fontaine–Laffaille, Fontaine–Messing, and Kato. Also included is an interesting example, generalized from a construction in Bhatt–Morrow–Scholze's work, which (1) illustrates some of our theoretical results being sharp, and (2) negates a question of Breuil.

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1. Introduction

Let $\mathcal{O}_K$ be a mixed characteristic DVR with perfect residue field k and fraction field K. Let $\mathcal{X}$ be a smooth proper (formal) scheme over $\mathcal{O}_K$, it is natural to ask how the geometry of $\mathcal{X}_k$ and $\mathcal{X}_K$ are related. Recall that proper base change theorem [Sta21, Tag 0GJ2] says that, for any prime ℓ, there is a specialization map

$$\mathrm{Sp}\colon \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell).$$

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The smooth base change theorem says [Sta21, Tag 0GKD] that the above map is an isomorphism for any $\ell \neq p$.

The lack of the smooth base change theorem when $\ell = p$ is related to many interesting "pathology" phenomena in p-adic cohomology theories. In this paper, we try to investigate these pathologies using the recent advances of prismatic cohomology theory.

The driving philosophy in this article is the following: recall in [BMS18], [BMS19], and [BS19], the authors attached a natural cohomology theory, known as the prismatic cohomology, to the mixed characteristic family $\mathcal{X}/\mathcal{O}_K$. This cohomology can be thought of as "the universal p-adic cohomology theory", therefore we expect certain well-defined piece inside prismatic cohomology to be "the universal source of pathology" in all p-adic cohomology theory. Before explicating the above, let us first say that the comparison between étale torsion and crystalline torsion as in [BMS18, Theorem 1.1.(ii)] serves as the initial inspiration. Now let us showcase two more such pathologies and state what our main theorem specializes to in these two cases.

Albanese and reduction. Let us assume, in addition to above, that $\mathcal{X}$ possesses an $\mathcal{O}_K$-point x. Associated with the pair $(\mathcal{X}, x)$ is a functorial map of abelian varieties $f\colon \mathrm{Alb}(\mathcal{X}_k) \to \mathcal{A}_k$ over k, where $\mathcal{A}$ is the Néron model of the Albanese of $(\mathcal{X}_K, x_K)$. The smooth and proper base change theorem tells us that f is a p-power isogeny. What can we say about the $\ker(f)$?

Theorem 1.1 (Corollary 4.6)**.** *Let e be the ramification index of $\mathcal{O}_K$.*

(1) If $e < p-1$ then the map $f\colon \mathrm{Alb}(\mathcal{X}_0) \to \mathrm{Alb}(X)_0$ is an isomorphism.
(2) If $e < 2(p-1)$ then $\ker(f)$ is p-torsion.
(3) If $e = p-1$ then $\ker(f)$ is p-torsion and of multiplicative type, hence must be a form of several copies of μ_p. Moreover there is a canonical injection of $\mathcal{O}_K$-modules

$$\mathbb{D}(\ker(f)) \otimes_k \big(\mathcal{O}_K/p\big) \hookrightarrow \mathrm{H}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

Here $\mathbb{D}(-)$ denotes the Dieudonné module of said finite flat group scheme. If one translates this result to a statement concerning maps between Picard schemes, then our result slightly refines an old result by Raynaud [Ray79, Thèoréme 4.1.3] in the setting of smooth central fiber, see Remark 4.8.

Kernel of specialization. The p-adic specialization map is not an isomorphism, as it is almost never going to be surjective, for the rank of the source is at most half of the rank of the target. One can still ask whether the p-adic specialization map is injective or not.

Theorem 1.2 (Corollary 4.15)**.** *Let e be the ramification index of $\mathcal{O}_K$, and let $i \in \mathbb{N}$. Consider the specialization map $\mathrm{Sp}^i\colon \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_p) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$.*

(1) If $e \cdot (i-1) < p-1$, then Sp^i is injective.
(2) If $e \cdot (i-1) < 2(p-1)$, then $\ker(\mathrm{Sp}^i)$ is annihilated by p^{i-1}.
(3) If $e \cdot (i-1) = p-1$, then $\ker(\mathrm{Sp}^i)$ is p-torsion, and there is a $\mathrm{Gal}(\overline{k}/k)$-equivariant injection:

$$\ker(\mathrm{Sp}^i) \otimes_{\mathbb{F}_p} \big(\mathcal{O}_K \otimes_W W(\overline{k})\big)/p \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}) \otimes_W W(\overline{k}).$$

The above two theorems are of similar shape, and that is because they are shadows of the same result concerning prismatic cohomology, which we explain next.

Prismatic input. Choose a uniformizer $\pi \in \mathcal{O}_K$, there is a canonical surjection $\mathfrak{S} := W(k)[\![u]\!] \twoheadrightarrow \mathcal{O}_K$ with kernel generated by the Eisenstein polynomial of π, which has degree given by the ramification index e. Let $\varphi_{\mathfrak{S}}$ be the endomorphism on $\mathfrak{S}$ which restricts to usual Frobenius on $W(k)$ and sends u to u^p. The triple $(\mathfrak{S}, (E), \varphi_{\mathfrak{S}})$ is known as the Breuil–Kisin prism associated with $(\mathcal{O}_K, \pi)$ [BS19, Example 1.3.(3)].

In [BMS19], and [BS19], the authors attached an $\mathfrak{S}$-perfect complex $\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S})$ with a Frobenius operator. Similar to the classical crystalline story, the Frobenius operator is also an isogeny. A concrete consequence of having an isogenous Frobenius map is that the torsion submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ is p-power torsion [BMS18, Proposition 4.3.(i)]. Hence the torsion must be supported on $\mathrm{Spec}(\mathfrak{S}/p)$, note that $\mathfrak{S}/p \cong k[\![u]\!]$ is a DVR. An upshot of the above discussion is that we have three descriptions of a submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$:

(1) the u^∞-torsion submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, from now on we denote it as $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$;

(2) the maximal finite length submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$; and
(3) the submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ supported at the closed point in $\mathrm{Spec}(\mathfrak{S})$.

To convey readers the above *is* the universal source of pathology in p-adic cohomology theory, let us exhibit the connection between u^∞-torsion and our results before.

Theorem 1.3.

(1) (Theorem 4.2) Concerning the natural map $f\colon \mathrm{Alb}(\mathcal{X}_0) \to \mathrm{Alb}(X)_0$, *we have a natural isomorphism of Dieudonné modules:*

$$\mathbb{D}(\ker(f))^{(-1)} \cong \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u],$$

where $(-)^{(-1)}$ *denote the Frobenius* untwist *and* $(-)[u]$ *denotes the* u*-torsion submodule.*

(2) (Theorem 4.14) As for the kernel of p*-adic specialization map, we have a natural isomorphism of* $\mathrm{Gal}(\overline{k}/k)$*-representations:*

$$\ker(\mathrm{Sp}^i) \cong \left(\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]/u \otimes_{W(k)} W(\overline{k})\right)^{\varphi=1}.$$

In view of aforementioned statements, the reader can probably guess what our main result, concerning the structure of u^∞-torsion in prismatic cohomology, should look like.

Theorem 1.4 (Theorem 3.3 and Corollary 3.19)**.** *Let us write* $\mathfrak{M}^i := \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$, *and write* $\mathrm{Ann}(-)$ *for the annihilator ideal of an* $\mathfrak{S}$*-module.*

(1) If $e\cdot(i-1) < p-1$, *then* $\mathfrak{M}^i = 0$.
(2) If $e\cdot(i-1) < 2(p-1)$, *then* $\mathrm{Ann}(\mathfrak{M}^i) + (u) \supset (p^{i-1}, u)$.
(3) If $e\cdot(i-1) = p-1$, *then* $\mathrm{Ann}(\mathfrak{M}^i) \supset (p,u)$. *Moreover the semi-linear Frobenius on* $\mathfrak{M}^i$ *is bijective, and there is a natural injection* $\mathfrak{M}^i \otimes_k \left(\mathcal{O}_K/p\right) \hookrightarrow \mathrm{H}^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Remark 1.5.

(1) We also prove the mod p^n analogs as well. As a consequence we obtain the following Corollary 3.5 concerning the shape of prismatic cohomology: Let i be an integer satisfying $e\cdot(i-1) < p-1$, then there exists a (non-canonical) isomorphism of $\mathfrak{S}$-modules:

$$\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}.$$

(2) Previously Min has obtained vanishing of $\mathfrak{M}^i$ with the assumption $e\cdot i < p-1$ [Min21, Theorem 0.1]. Let us briefly explain the appearance of $(i-1)$ in our result, which might seem odd at first glance. It is due to the fact that the prismatic Verschiebung operator V_i becomes canonically divisible by E when restricted to the p^∞-torsion submodule or the u^∞-torsion submodule, and these submodules with the usual prismatic Frobenius and the "divided Verschiebung" is canonically a (generalized) Kisin module of height $(i-1)$ instead of i. For more details, see Corollary 3.13.
(3) One may ask if there can be a better trick/argument showing better bounds on vanishing of u-torsion. Later on we shall explain a generalization of a construction in [BMS18, Subsection 2.1] with u-torsion in cohomological degree 2 and ramification index $p-1$. Hence our result is actually sharp in terms of largest $e\cdot(i-1)$ allowed.

Special fiber telling Hodge numbers of the generic fiber. As a third geometric application of our result, we revisit the question discussed in [Li20]: what mild condition on $\mathcal{X}$ guarantees that the Hodge numbers of the generic fibre X can be read off from the special fibre $\mathcal{X}_0$? In loc. cit. the first named author obtained a result along this line, with technical input of prismatic cohomology and the structural result in [Min21]. However it was noted that the results in loc. cit. are not optimal already in the unramified case, when compared with what one got from results by Fontaine–Messing, Kato, and Wintenberger. We analyzed the situation and concluded that it is because we lack knowledge of the shape of u^∞-torsion in prismatic cohomology in the boundary degree. This paper is partially motivated by the hope to improve results in [Li20], and our improvement is:

Theorem 1.6 (Theorem 4.17, Improvement of [Li20, Theorem 1.1])**.** *Let* $\mathcal{X}$ *be a smooth proper* p*-adic formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$ *of ramification index* e. *Let* T *be the largest integer such that* $e\cdot(T-1) \le p-1$.

(1) Assume there is a lift of $\mathcal{X}$ to $\mathfrak{S}/(E^2)$, then for all i,j satisfying $i+j<T$, we have equalities

$$h^{i,j}(X)=\mathfrak{h}^{i,j}(\mathcal{X}_0)$$

where the latter denotes virtual Hodge numbers of $\mathcal{X}_0$, defined as in [Li20, Definition 3.1].

(2) Assume furthermore that $e\cdot(\dim\mathcal{X}_0-1)\le p-1$. Then the special fibre $\mathcal{X}_0$ knows the Hodge numbers of the rigid generic fibre X.

Along the way, we also improve the results in [Li20] concerning the integral Hodge–de Rham spectral sequence (see Theorem 4.18), as well as obtain a curious degeneration statement of the "Nygaard–Prism" spectral sequence (see Theorem 4.22) in the unramified case.

Application to integral p-adic Hodge theory. It is a central theme in integral p-adic Hodge theory to understand Galois representations such as $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}},\mathbb{Z}_p)$ in terms of linear algebraic data such as certain crystalline cohomology of $\mathcal{X}$ together with natural structures. The first result along such lines is that of Fontaine–Messing [FM87] and Kato [Kat87], which treats the case of $e=1$ (namely unramified base) and $i<p-1$.[1] Later on Breuil [Bre98] generalized the above to semistable $\mathcal{X}$, whereas Faltings [Fal99] studied the analogue for p-divisible groups allowing arbitrary ramification index e. A few years later, Caruso [Car08] made progress allowing $e>1$ as long as $e\cdot(i+1)<p-1$,[2]

To our interest is Breuil's question [Bre02, Question 4.1]:

Question 1.7. Assuming $i<p-1$ and let S be the p-adic divided power envelope of $\mathfrak{S}\twoheadrightarrow\mathcal{O}_K$, then the (torsion-free) crystalline cohomology $\mathrm{H}^i_{\text{crys}}(\mathcal{X}/S)/\text{tors}$ together with its natural structure (such as divided Frobenius operator, filtration and connection) should be a "strongly divisible lattice" and "corresponds" to the Galois representation $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}},\mathbb{Z}_p)$.

All works mentioned above can be thought of as solving various special cases of the above question. In [LL20, Theorem 7.22], a connection with u-torsion in prismatic cohomology is observed: We showed that, fix an $i<p-1$ and a smooth proper formal scheme $\mathcal{X}/\mathcal{O}_K$, the mod p^n analogue of the above question has a positive answer in degree i if and only if both of i-th and $(i+1)$-st mod p^n prismatic cohomology of $\mathcal{X}/\mathfrak{S}$ are u-torsion free. In loc. cit., we then used Caruso's result on the mod p analog as a starting point to do an induction to show the vanishing as in Theorem 1.4 (1) and Remark 1.5 (1), which in turn implies the mod p^n analog of Breuil's question for all n and $e\cdot i<p-1$, see [LL20, Corollary 7.25]. In particular this gives an affirmative answer to Breuil's original question when $e\cdot i<p-1$. In this paper, the aforementioned vanishing of u-torsion is easily deduced, hence gives a "shortcut" to the above result bypassing Caruso's work.

In private communications with Breuil, we were encouraged to study his question beyond the above bound. To our surprise, we discovered that the construction in [BMS18, Subsection 2.1] can be generalized to a counterexample with $e=p-1$ and $i=1$ to Breuil's question, see Example 1.10. Note that in this example, we have $e\cdot i=p-1$, hence our previous result was actually sharp.

The other extreme of (e,i) with $e\cdot i=p-1$ is $e=1,i=p-1$. In this case, Fontaine–Messing [FM87] and Kato [Kat87] showed that the crystalline cohomology $\mathrm{H}^{p-1}_{\text{crys}}(\mathcal{X}_n/W_n)$ together with its natural structure is still a Fontaine–Laffaille module, which according to [FL82] can be attached a Galois representation $\rho^{p-1}_{n,\text{FL}}$. It is only natural to ask:

Question 1.8. What is the relation between $\rho^{p-1}_{n,\text{FL}}$ and $\mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\overline{K}},\mathbb{Z}/p^n)$?

Although we have not found any discussion on this question, it seems consensus among experts that these two Galois representations are different. We are not aware of any particular expectation made in the past. Our entire Section 5 is more or less devoted to this question, and we arrive at the following statement.

Theorem 1.9 (Theorem 5.28)**.** *There exists a natural map $\eta\colon\mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\mathbf{C}},\mathbb{Z}/p^n\mathbb{Z})(p-1)\to\rho^{p-1}_{n,\text{FL}}$ of G_K-representations so that* $\ker(\eta)$ *is an unramified representation of G_K killed by p, and* $\operatorname{coker}(\eta)$ *sits in a natural exact sequence* $0\to\ker(\eta)\to\operatorname{coker}(\eta)\to\ker(\mathrm{Sp}^{p-1}_n)$.

[1] See also [AMMN21] for an approach of different flavor.

[2] For the mod p analogue, Caruso's work even allows $e\cdot i<p-1$.

Here Sp_n^{p-1} denotes the specialization map of mod p^n étale cohomology in degree $p-1$, which is also known to be an unramified G_K-representation killed by p, see Corollary 4.15 (3). The appearance of $\ker(\eta)$ is due to the defect of a key functor in integral p-adic Hodge theory, which is well-known to experts; whereas the potential u-torsion in degree p of mod p^n prismatic cohomology of $\mathcal{X}$ is solely responsible for the appearance of $\ker(\mathrm{Sp}_n^{p-1})$.

Example and open questions. Now let us discuss an interesting example, generalized from [BMS18, Subsection 2.1].

Example 1.10. Let $\mathcal{E}/W(\overline{k})$ be the canonical lift of an ordinary elliptic curve over an algebraically closed field $\overline{k}$ of characteristic p. Fix an $n \in \mathbb{N}$ and let $\mathcal{O}_K \coloneqq W(\overline{k})[\zeta_{p^n}]$. Over $\mathcal{O}_K$ we have a tautological map of group schemes $\chi\colon \mathbb{Z}/p^n \to \mu_{p^n}$ sending 1 to ζ_{p^n}.

With the above notation, we consider the following smooth proper Deligne–Mumford stack $\mathcal{X} \coloneqq [\mathcal{E}_{\mathcal{O}_K}/(\mathbb{Z}/p^n)]$ where the action of $\mathbb{Z}/p^n$ is via the character χ and the embedding $\mu_{p^n} \subset \mathcal{E}[p^n]$ (as $\mathcal{E}$ is the canonical lift). Note that its special fiber is $\mathcal{E}_{\overline{k}} \times B(\mathbb{Z}/p^n)$ and its generic fiber is an elliptic curve $\mathcal{E}'_K \coloneqq \left(\mathcal{E}_{\mathcal{O}_K}/\mu_{p^n}\right)_K$. In view of the pathologies discussed in the beginning of the introduction, let us record some facts concerning this example:

- The Albanese map has Néron model given by the "further quotient" map: $\mathcal{X} \to \mathcal{E}' \coloneqq \mathcal{E}_{\mathcal{O}_K}/\mu_{p^n}$, and the special fiber of this map factors as $\mathcal{E}_{\overline{k}} \times B(\mathbb{Z}/p^n) \to \mathcal{E}_{\overline{k}} \xrightarrow{f} \mathcal{E}_{\overline{k}}/\mu_{p^n}$. Here $\mathcal{E}'$ is abstractly isomorphic to $\mathcal{E}_{\mathcal{O}_K}$ since $\mathcal{E}$ was chosen to be the canonical lift. Note that $\ker(f) = \mu_{p^n}$.
- The fundamental group of $\mathcal{X}_{\overline{k}}$ is abelian, with torsion given by $\mathbb{Z}/p^n$ due to the factor of $B(\mathbb{Z}/p^n)$. By universal coefficient theorem, we have $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_p)_{\mathrm{tors}} \cong \mathbb{Z}/p^n$ whereas $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$ is torsion-free. Hence we have $\ker(\mathrm{Sp}^2) = \mathbb{Z}/p^n$.
- One can go through the Leray spectral sequence for the cover $\mathcal{E} \to \mathcal{X}$ to compute the prismatic cohomology of $\mathcal{X}/\mathfrak{S}$. The most relevant computation is: $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty] \cong \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n)$.
- Finally we compute the crystalline cohomology of $\mathcal{X}/S$ and to our surprise we have $\mathrm{H}^1(\mathcal{X}/S) \cong S \oplus J$ where J is the ideal
$$\{x \in S \mid p^n \text{ divides } x \cdot ((u+1)^{p^n} - 1)\}.$$
In particular it is torsion-free of rank 2 yet not free. This gives a counterexample to Question 1.7.

By standard approximation technique, one can cook up schematic examples having all the above features. When $n = 1$, we have $e = p-1$, therefore our aforesaid results (which was only stated and proved for formal schemes) are sharp. For more details, see Section 6.

Combining a generalized version of the above construction with our Theorem 1.1, we get a geometric proof of Raynaud's theorem [Ray74, Théorème 3.3.3] on prolongations of finite flat commutative group schemes over mixed characteristic DVR, see Section 6.1.

Finally, let us end the introduction with two natural questions awaiting explorations. We consider them to be the next step in understanding pathological torsion in p-adic cohomology theory.

Question 1.11. Is there a smooth proper (formal) scheme $\mathcal{X}$ over an unramified base W which has u-torsion in its p-th prismatic cohomology? Note that p is the smallest possible cohomological degree according to our result, and when $p = 2$ this is achieved by the above example.

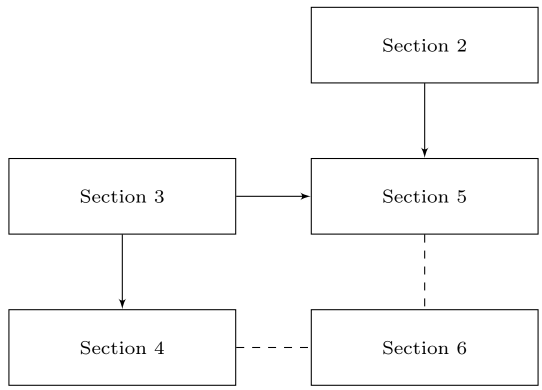
Question 1.12 (see Question 3.7)**.** Recall $\mathfrak{M}^i \coloneqq \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta)[u^\infty]$.

(1) Let β be the smallest exponent such that $p^\beta \in \mathrm{Ann}(\mathfrak{M}^i)$, let γ be the exponent such that $\mathrm{Ann}(\mathfrak{M}^i) + (u) = (u, p^\gamma)$. Is there a bound on β and γ in terms of e and i?
(2) In light of the above example, we guess β and/or γ are bounded above by $\log_p(\frac{e\cdot(i-1)}{p-1}) + 1$ when p is odd.

Remark 1.13. Confirming the above guess will give us results along the following line: If $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)$ has torsion *not* annihilated by p^N, then $\mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$ has torsion *not* annihilated by $p^{N-c(e,i)}$ with $c(e,i)$ being some constants depending only on e and i. Note that this would be a relation between torsion in étale and crystalline cohomology "converse" to the one established in [BMS18, Theorem 1.1.(ii)]. When $e \cdot i < 2(p-1)$, our Theorem 1.4.(2) can be translated to such a statement. Since our Theorem 1.4.(2) does not seem to be optimal, we do not pursue that direction in this paper.

The logic between each section is as below.

Leitfaden



Notation and Conventions. Let k be a perfect field of characteristic $p > 0$ with $W = W(k)$ its Witt ring. Let K be a totally ramified degree e finite extension of $W(k)[1/p]$, let $\mathcal{O}_K$ be its ring of integers. Choose a uniformizer $\pi \in \mathcal{O}_K$ whose Eisenstein polynomial we denote by E with $E(0) = a_0 p$, we get a surjection $\mathfrak{S} := W[\![u]\!] \twoheadrightarrow \mathcal{O}_K$ sending u to π. We equip $\mathfrak{S}$ the δ-structure with $\varphi_{\mathfrak{S}}(u) = u^p$. The pair $(\mathfrak{S}, (E))$ is the so-called Breuil–Kisin prism, see [BS19, Example 1.3.(3)]. Denote the p-adic divided power envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ by S.

We always use C and its cousins like $C^\flat$ or A_{inf}, to denote the usual construction associated with the completion of an algebraic closure $\overline{K}$ of K in p-adic Hodge theory. We use $G_K := \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group. Similarly, $G_k := \mathrm{Gal}(\overline{k}/k)$.

We use $\mathcal{X}$ to denote a smooth proper p-adic formal scheme on $\mathrm{Spf}(\mathcal{O}_K)$, use $\mathcal{X}_0$ to denote its reduction mod π and use X to denote its rigid generic fiber.

On $(\mathcal{O}_K)_{\mathrm{qSyn}}$ we have the sheaf Δ given by (left Kan extended) prismatic cohomology relative to $\mathfrak{S}$. We use $\Delta^{(1)}$ to denote its $\varphi_{\mathfrak{S}}$ twist, this sheaf of Frobenius-twisted prismatic cohomology admits a decreasing filtration called the Nygaard filtration, see [BS19, Section 15], which shall be denoted by $\mathrm{Fil}^\bullet_{\mathrm{N}}$. Let us note that $\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X}, \Delta)$ and $\varphi_{\mathfrak{S}}^* \mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)})$.

For any $n \in \mathbb{N} \cup \{\infty\}$, we use subscript $(-)_n$ to denote the derived mod p^n of a quasi-syntomic sheaf, e.g. $\mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)}) := \mathrm{R\Gamma}_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}/p^n)$.

In this paper we only consider relative prismatic cohomology, and hopefully readers will not confuse our notation with the absolute prismatic cohomology developed in [BL22].

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2. Various modules and their Galois representations

In this section, we discuss 3 types of Frobenius modules: Kisin modules, Breuil modules and Fontaine–Laffaille modules, and their associated Galois representations. Roughly speaking, various cohomology discussed in this paper will have these structures and functors to Galois representations just model comparison to étale cohomology. The major difference between the current work and [LL20] is that we now focus on the boundary case $eh = p-1$. So it is necessary to discuss *nilpotent* objects for Fontaine–Laffaille modules and Breuil modules when $e = 1$ and $h = p-1$.

2.1. **Kisin modules.** We review (generalized) Kisin modules from [LL20, §6.1]. Let $(\mathfrak{S}, E(u))$ be the Breuil–Kisin prism over $\mathcal{O}_K$ with $d = E(u) = E$ the Eisenstein polynomial of a fixed uniformizer $\pi \in \mathcal{O}_K$. A φ-module $\mathfrak{M}$ over $\mathfrak{S}$ is an $\mathfrak{S}$-module $\mathfrak{M}$ together $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. Write $\varphi^*\mathfrak{M} = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Note that $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an $\mathfrak{S}$-linear map. A *(generalized) Kisin module $\mathfrak{M}$ of height h* is a φ-module $\mathfrak{M}$ of finite $\mathfrak{S}$-type together with an $\mathfrak{S}$-linear map $\psi : \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\psi \circ (1 \otimes \varphi) = E^h \operatorname{id}_{\varphi^*\mathfrak{M}}$ and $(1 \otimes \varphi) \circ \psi = E^h \operatorname{id}_{\mathfrak{M}}$. The map between generalized Kisin modules is $\mathfrak{S}$-linear map that compatible with φ and ψ. We denote by $\mathrm{Mod}_{\mathfrak{S}}^{\varphi, h}$ the category of (generalized) Kisin module of height h. As explained in [LL20], the main difference between generalized Kisin modules and classical theory Kisin modules is that the classical theory only discuss the situation that $\mathfrak{M}$ has no u-torsion, while Kisin module from prismatic cohomology could have u-torsion in general. In the following, when we need to restrict to the classical theory, we will call $\mathfrak{M}$ either *classical* or *u-torsion free*. Let $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi, h, \mathrm{c}}$ denote the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi, h}$ consists of classical Kisin modules of height h and killed by p^n for some $n \in \mathbb{N}$.

Now we review some technologies to deal with classical Kisin modules on boundary case and extends them to generalized Kisin modules. Following [Kis09, (1.2.10)], [Gao17, §2.1], we call a φ-module $\mathfrak{M}$ *multiplicative* (resp. *nilpotent*) if $(1 \otimes \varphi) : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is surjective (resp. if $\lim\limits_{n \to \infty} \varphi^n(x) = 0, \ \forall x \in \mathfrak{M}$).

Remark 2.1. In [Kis09, (1.2.10)] and [Gao17, §2.1], the authors define *multiplicative* to mean $(1 \otimes \varphi) : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is *bijective*. For classical Kisin module $\mathfrak{M}$ these two concepts are the same as $1 \otimes \varphi$ is always injective. But for generalized Kisin modules, as u-torsion exists, bijection of $1 \otimes \varphi$ is too restrictive. For example, $\mathfrak{S}/(p, u)\mathfrak{S}$ with the usual Frobenius is multiplicative but $1 \otimes \varphi$ is not injective.

Let $\mathfrak{M}$ be a φ-module over $\mathfrak{S}$ of finite $\mathfrak{S}$-type. Set $M \coloneqq \mathfrak{M}/u\mathfrak{M}$ and write $q \colon \mathfrak{M} \to M = \mathfrak{M}/u\mathfrak{M}$. By Fitting lemma, we have $M = M^{\mathrm{m}} \oplus M^{\mathrm{n}}$ where φ is bijective on M^{m} and nilpotent on M^{n}.

Lemma 2.2. *Notations as the above, there exists a unique $W(k)$-linear section $[\cdot] : M^{\mathrm{m}} \to \mathfrak{M}$ so that $[\cdot]$ is φ-equivariant and $q \circ [\cdot] = \operatorname{id}|_{M^{\mathrm{m}}}$.*

Proof. Pick any $\overline{x} \in M^{\mathrm{m}}$, since φ on M^{m} is bijective, there exists unique $\overline{x}_n \in M$ so that $\varphi^n(\overline{x}_n) = \overline{x}$. Select $x_n \in \mathfrak{M}$ a lift of $\overline{x}_n$ and define $[\overline{x}] := \lim\limits_{n \to \infty} \varphi^n(x_n)$. We first check that $\varphi^n(x_n)$ converges to an $x \in \mathfrak{M}$ so that $q(x) = \overline{x}$. Indeed, since $\varphi(\overline{x}_{n+1}) = \overline{x}_n$, $\varphi(x_{n+1}) - x_n = uy_n$ with $y_n \in \mathfrak{M}$. So $\varphi^{n+1}(x_{n+1}) - \varphi^n(x_n) = \varphi^n(u)\varphi^n(y)$ and hence $\varphi^n(x_n)$ converges to a $x \in \mathfrak{M}$ and clearly $q(x) = \overline{x}$. Suppose that $x'_n \in \mathfrak{M}$ is another lift of $\overline{x}_n$, then $x'_n - x_n = uz_n$ with $z_n \in \mathfrak{M}_n$. Then $\varphi^n(x'_n) - \varphi^n(x_n) = u^{p^n}\varphi^n(z_n)$. So $\{\varphi^n(x'_n)\}$ also converges to x. This implies that $x = [\overline{x}]$ does not depend on the choice of lift x_n of $\overline{x}_n = \varphi^{-n}(\overline{x})$. Hence the section $[\cdot] : M^{\mathrm{m}} \to \mathfrak{M}$ is well-defined and satisfies $q \circ [\cdot] = \operatorname{id}_{M^{\mathrm{m}}}$. For any $a \in W(k)$, it is clear that $a[\overline{x}] = [a\overline{x}]$ from construction of $[\overline{x}]$. So $[\cdot]$ is $W(k)$-linear. If $\overline{y} = \varphi(\overline{x})$ then $\varphi(x_n)$ is a lift of $\varphi^{-n}(y)$ and $[\overline{y}] = \lim\limits_{n \to \infty} \varphi^{n+1}(x_n) = \varphi(x) = \varphi([\overline{x}])$. So $[\cdot]$ is φ-equivariant. Finally, suppose there is another sections $[\cdot]' : M^{\mathrm{m}} \to \mathfrak{M}$. Then $[\overline{x}] - [\overline{x}]' \in u\mathfrak{M}$ for any $\overline{x} \in M^{\mathrm{m}}$. Then $[\overline{x}] - [\overline{x}]' = \varphi^n([\overline{x}_n] - [\overline{x}_n]') \in u^{p^n}\mathfrak{M}$. This forces that $[\overline{x}] = [\overline{x}]'$. ☐

Lemma 2.3. *Let $\mathfrak{M}$ be a φ-module with finite $\mathfrak{S}$-type. Then there exists an exact sequence of φ-modules*

$$0 \longrightarrow \mathfrak{M}^{\mathrm{m}} \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}^{\mathrm{n}} \longrightarrow 0 \tag{2.4}$$

so that $\mathfrak{M}^{\mathrm{m}}$ is multiplicative and $\mathfrak{M}^{\mathrm{n}}$ has no nontrivial multiplicative submodule. Furthermore, the above exact sequence is functorial for $\mathfrak{M}$, and if $\mathfrak{M}$ is in $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi, h, \mathrm{c}}$ then so are $\mathfrak{M}^{\mathrm{m}}$ and $\mathfrak{M}^{\mathrm{n}}$.

Proof. Note that [Kis09, Prop. (1.2.11)] has treated the situation that $\mathfrak{M}$ has no u-torsion but our idea here is slightly different. By the above lemma, we can set $\mathfrak{M}^{\mathrm{m}}$ to be the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by $[M^{\mathrm{m}}]$ and $\mathfrak{M}^{\mathrm{n}} := \mathfrak{M}/\mathfrak{M}^{\mathrm{m}}$. Clearly, $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{\mathrm{m}} \to \mathfrak{M}^{\mathrm{m}}$ is surjective. Consider the right exact sequence $\mathfrak{S} \otimes_{W(k)} [M] \to \mathfrak{M} \to \mathfrak{M}^{\mathrm{n}} \to 0$. By modulo u, we have the right exact (indeed exact) sequence $M^{\mathrm{m}} \to M \to \mathfrak{M}^{\mathrm{n}}/u\mathfrak{M}^{\mathrm{n}} \to 0$. So $\mathfrak{M}^{\mathrm{n}}/u\mathfrak{M}^{\mathrm{n}} \simeq M^{\mathrm{n}}$ and also forces $\mathfrak{M}^{\mathrm{m}}/u\mathfrak{M}^{\mathrm{m}} = M^{\mathrm{m}}$. Hence φ on $\mathfrak{M}^{\mathrm{n}}$ is topologically nilpotent as φ on M^{n} is nilpotent, thus $\mathfrak{M}^{\mathrm{n}}$ can not have nontrivial multiplicative submodule. So we obtain exact sequence (2.4) which is functorial for $\mathfrak{M}$ because $[\cdot]$ is clearly functorial for $\mathfrak{M}$ by the above lemma.

If $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ then $\mathfrak{M}^{\mathrm{m}}$ has no u-torsion. Note that the exact sequence (2.4) modulo u becomes the exact sequence $0 \to M^{\mathrm{m}} \to M \to M^{\mathrm{n}} \to 0$. Then $\mathfrak{M}^{\mathrm{n}}$ can not have u-torsion as $\mathfrak{M}$ has no u-torsion. Hence both $\mathfrak{M}^{\mathrm{m}}$ and $\mathfrak{M}^{\mathrm{n}}$ have no u-torsion. Then both $\mathfrak{M}^{\mathrm{m}}$ and $\mathfrak{M}^{\mathrm{n}}$ have E-height h by [Fon90, Prop. B 1.3.5] as required. $\square$

But for a generalized Kisin module $\mathfrak{M}$ with height h, it is unclear if we can define $\psi : \mathfrak{M}^{\mathrm{m}} \to \varphi^*\mathfrak{M}^{\mathrm{m}}$ so that $\mathfrak{M}^{\mathrm{m}}$ has height h. Luckily, we will not need such a statement.

Let $M[p^n]$ denote the p^n-torsion in M, for later application, we need the following two statements.

Lemma 2.5. *Let M be a finitely generated $\mathfrak{S}$-module. Assume M/p^nM are u-torsion free for all $n > 0$. Then $M/(M[p^n] + pM)$ are also u-torsion free for all $n > 0$.*

Proof. Suppose $x \in M$ is a lift of a u-torsion in $M/(M[p^n] + pM)$, hence satisfies $u \cdot x = y + p \cdot z$ for some $y \in M[p^n]$ and $z \in M$. Multiply the equation by p^n, we get $u \cdot p^n \cdot x = p^{n+1} \cdot z$. As $M/p^{n+1}M$ also has no u-torsion by assumption, we see that $p^n \cdot x = p^{n+1} \cdot \widetilde{z}$ for some $\widetilde{z} \in M$. Write $x = (x - p \cdot \widetilde{z}) + p \cdot \widetilde{z}$ shows that in fact $x \in M[p^n] + pM$, as required. $\square$

Proposition 2.6. *Let M be a finitely generated generalized Breuil–Kisin module. Assume M/p^nM are u-torsion free for all $n > 0$. Then there exists a $\mathbb{Z}_p$-module N and an isomorphism of $\mathfrak{S}$-modules $M \simeq N \otimes_{\mathbb{Z}_p} \mathfrak{S}$.*

Proof. First let us treat the case when M is torsion. In this case M is killed by a power of p, see [BMS18, Proposition 4.3.(i)]. Denote $\mathrm{Im}(M \xrightarrow{p} M) = pM =: M_1$. We claim M_1/p^nM_1 are also u-torsion free for all $n > 0$. Granting this claim, by induction on the exponent of p annihilating M, we know M_1 satisfies the conclusion. Here, for the starting point of induction, we used the fact that a finitely generated $\mathfrak{S}/p$-module is u-torsion free if and only if it is free. Then by [Min21, Lemma 5.9], we get the conclusion for M.

We now verify the claim. Applying the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\cdot p^n"] & M \arrow[r] \arrow[d, "\cdot p^n"] & M/M_1 \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M/M_1 \arrow[r] & 0 \end{tikzcd}$$

yields an exact sequence

$$0 \to M/(M[p^n] + pM) \to M_1/p^nM_1 \to M/p^nM.$$

Here $M[p^n]$ denotes the p^n-torsion in M. Since M/p^nM has no u-torsion by assumption, it suffices to show the same for $M/(M[p^n] + pM)$. Applying Lemma 2.5 gives the claim.

Next we turn to the general case. By [BMS18, Proposition 4.3], we have two short exact sequences of generalized Breuil–Kisin modules

$$0 \to M_{\mathrm{tor}} \to M \to M_{\mathrm{tf}} \to 0$$

and

$$0 \to M_{\mathrm{tf}} \to M_{\mathrm{fr}} \to M_0 \to 0.$$

Here M_{tor} is the torsion submodule, M_{tf} is the torsion free quotient, M_{fr} is the reflexive hull of M (which is free as $\mathfrak{S}$ is a 2-dimensional regular Noetherian domain), and M_0 has finite length. The first sequence implies that M_{tor}/p^n injects into M/p^n, therefore M_{tor} satisfies the assumption. Since we have treated the torsion case, we see that M_{tor} satisfies the conclusion. Now we claim M_0 vanishes. This immediately implies that $M_{\mathrm{tf}} = M_{\mathrm{fr}}$ is free, hence the first sequence splits, and $M = M_{\mathrm{tor}} \oplus M_{\mathrm{tf}}$ has shape of a $\mathbb{Z}_p$-module.

Finally let us justify the claim that $M_0 = 0$. Take the second sequence above, derived modulo p gives an inclusion $M_0[p] \subset M_{\mathrm{tf}}/p$. Since M_0 has finite length, we see that $M_0[p]$ must be u^∞-torsion. If we can show that M_{tf}/p is u-torsion free, then we get $M_0[p] = 0$ which implies $M_0 = 0$ as it must be p^∞-torsion. We now reduce ourselves to showing $M/(M_{\mathrm{tor}} + p \cdot M)$ is u-torsion free. Since $M_{\mathrm{tor}} = M[p^n]$ for sufficiently large n, we finish the proof by appealing to Lemma 2.5. $\square$

2.2. **Breuil modules.** Fix $0 \le h \le p-1$. Let S be the p-adically completed PD-envelope of $\theta : \mathfrak{S} \twoheadrightarrow \mathcal{O}_K, u \mapsto \pi$, and for $i \ge 1$ write $\mathrm{Fil}^i S \subseteq S$ for the (closure of the) ideal generated by $\{\gamma_n(E) = E^n/n!\}_{n \ge i}$. For $i \le p-1$, one has $\varphi(\mathrm{Fil}^i S) \subseteq p^i S$, so we may define $\varphi_i : \mathrm{Fil}^i S \to S$ where $\varphi_i := p^{-i}\varphi$. We have $c_1 := \varphi(E(u))/p \in S^\times$. Note that $S \subset K_0[\![u]\!]$. Define $I_+ := S \cap uK_0[\![u]\!]$. Clearly $S/I_+ = W(k)$. Let $S_n := S/p^n S$. Let ${}^\sim\mathrm{Mod}_S^{\varphi,h}$ denote the category whose objects are triples $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$, consisting of

(1) two S-modules $\mathcal{M}$ and $\mathrm{Fil}^h \mathcal{M}$;
(2) an S-module map $\iota : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ whose image contains $\mathrm{Fil}^h S \cdot \mathcal{M}$; and
(3) a φ-semi-linear map $\varphi_h : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^h S$ and $x \in \mathcal{M}$ we have

$$\varphi_h(sx) = (c_1)^{-h}\varphi_h(s)\varphi_h(E(u)^h x).$$

Morphisms are given by S-linear maps compatible with ι's and commuting with φ_h. Let ${}'\mathrm{Mod}_S^{\varphi,h}$ denote the full subcategory of ${}^\sim\mathrm{Mod}_S^{\varphi,h}$ whose objects $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$ satisfy

(1) ι is injective so that $\mathrm{Fil}^h \mathcal{M}$ is regarded as a submodule of $\mathcal{M}$.
(2) $\varphi_h(\mathrm{Fil}^h \mathcal{M})$ generates $\mathcal{M}$ as S-modules.

A sequence is defined to be *short exact* if it is short exact as a sequence of S-module, and induces a short exact sequence on Fil^h's. Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ denote the full subcategory of ${}'\mathrm{Mod}_S^{\varphi,h}$ so that the underlying module $\mathcal{M}$ is killed by a p-power and the triple $\mathcal{M}$ can be a written as successive extensions of triples $\mathcal{M}_i$ in ${}'\mathrm{Mod}_S^{\varphi,h}$ with each underlying module $\mathcal{M}_i \simeq \bigoplus_{\mathrm{finite}} S_1$.

Let $\nabla : S \to S$ be $W(k)$-linear continuous derivation so that $\nabla(u) = 1$. Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$ denote the category of the object $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h, \nabla)$ where $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$ is an object in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and ∇ is $W(k)$-linear morphism $\nabla : \mathcal{M} \to \mathcal{M}$ such that :

(1) for all $s \in S$ and $x \in \mathcal{M}$, $\nabla(sx) = \nabla(s)x + s\nabla(x)$.
(2) $E\nabla(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$.
(3) the following diagram commutes:

$$\begin{tikzcd} \mathrm{Fil}^h \mathcal{M} \arrow[r, "\varphi_h"] \arrow[d, "E(u)\nabla"'] & \mathcal{M} \arrow[d, "c_1\nabla"] \\ \mathrm{Fil}^h \mathcal{M} \arrow[r, "u^{p-1}\varphi_h"] & \mathcal{M} \end{tikzcd} \tag{2.7}$$

An object $\mathcal{M}$ in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ is called a (torsion) *Breuil module*.

Now let us recall the relation of classical torsion Kisin modules and objects in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. For each such $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$, we construct an object $\mathcal{M} := \underline{\mathcal{M}}(\mathfrak{M}) \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ as the following: $\mathcal{M} := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and

$$\mathrm{Fil}^h \mathcal{M} := \{x \in \mathcal{M} | (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M}\};$$

and $\varphi_h : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ is defined as the composite of following map

$$\mathrm{Fil}^h \mathcal{M} \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_h\otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M} .$$

For any $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, define a semi-linear $\varphi : \mathcal{M} \to \mathcal{M}$ by $\varphi(x) = (c_1)^{-h}\varphi_h(E^h x)$. Similar to the situation of Kisin module, we say $\mathcal{M}$ is *multiplicative* (resp. *nilpotent*) if $1 \otimes \varphi : S \otimes_{\varphi,S} \mathcal{M} \to \mathcal{M}$ is surjective (resp. $\lim_{n\to\infty} \varphi^n(x) = 0,\ \forall x \in \mathcal{M}$). Clearly if $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ is multiplicative (resp. nilpotent) then so is $\underline{\mathcal{M}}(\mathfrak{M})$.

Remark 2.8. Here our definition of multiplicative is different from that in [Gao17, Def. 2.2.2] where $\mathcal{M}$ is called multiplicative if $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\mathcal{M}$. Indeed these two definitions are equivalent. Suppose that $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\mathcal{M}$. Since $\varphi_h(ax) = \varphi_h(a)\varphi(x)$ for any $a \in \mathrm{Fil}^h S$ and $x \in \mathcal{M}$, $\{\varphi(x) = c_1^{-h}\varphi_h(E^h x)\}$ and $\{\varphi_h(\mathrm{Fil}^h S\mathcal{M})\}$ generates the same subsets in $\mathcal{M}$. This implies that $\varphi(\mathcal{M})$ generates $\mathcal{M}$. Conversely, suppose that $\varphi(\mathcal{M})$ generates $\mathcal{M}$. To show that $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\mathcal{M}$, we can reduce to the case that $\mathcal{M}$ is finite S_1-free by dévissage. See the last part of proof of Lemma 2.9.

Lemma 2.9. *For any object $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, there exists a functorial exact sequence*

$$0 \longrightarrow \mathcal{M}^{\mathrm{m}} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}^{\mathrm{n}} \longrightarrow 0 \tag{2.10}$$

with $\mathcal{M}^{\mathrm{m}}$ a multiplicative submodule of $\mathcal{M}$ and $\mathcal{M}^{\mathrm{n}}$ being nilpotent.

Proof. Recall $I_+ = S \cap uK_0[\![u]\!]$, $S/I_+ \simeq W(k)$ and $\varphi(x) = c_1^{-h}\varphi_h(E^h x)$. Write $S_n := S/p^n S$ and assume that $\mathcal{M}$ is an S_n-module. We claim Lemma 2.2 still holds by replacing $\mathfrak{M}$ by $\mathcal{M}$, $M = \mathcal{M}/I_+$ and $q : \mathcal{M} \to M = \mathcal{M}/I_+$. Indeed, the same proof goes through because $\varphi^\ell(I_+) = 0$ in S_n for sufficient large ℓ. Now we can set $\mathcal{M}^{\mathrm{m}}$ be S-submodule of $\mathcal{M}$ generated by $[M^{\mathrm{m}}]$ and $\mathcal{M}^{\mathrm{n}} := \mathcal{M}/\mathcal{M}^{\mathrm{m}}$. Using the same argument as in Lemma 2.3, the right exact sequence $S \otimes_{W(k)} [M^{\mathrm{m}}] \to \mathcal{M} \to \mathcal{M}^{\mathrm{n}} \to 0$ modulo I_+ becomes an exact sequence $0 \to M^{\mathrm{m}} \to M \to M^{\mathrm{n}} \to 0$. This forces to that $\mathcal{M}^{\mathrm{m}}/I_+ = M^{\mathrm{m}}$ and $\mathcal{M}^{\mathrm{n}}/I_+ = M^{\mathrm{n}}$. Set $\mathrm{Fil}^h \mathcal{M}^m = \mathrm{Fil}^h S \cdot \mathcal{M}^{\mathrm{m}}$ and $\mathrm{Fil}^h \mathcal{M}^{\mathrm{n}} = \mathrm{Fil}^h \mathcal{M}/\mathrm{Fil}^h \mathcal{M}^{\mathrm{m}}$. It is clear that $\varphi_h : \mathrm{Fil}^h \mathcal{M}^{\mathrm{m}} \to \mathcal{M}^{\mathrm{m}}$ and $\varphi_h : \mathrm{Fil}^h \mathcal{M}^{\mathrm{n}} \to \mathcal{M}^{\mathrm{n}}$ are well defined. So we obtain an exact sequence $0 \to \mathcal{M}^{\mathrm{m}} \to \mathcal{M} \to \mathcal{M}^{\mathrm{n}} \to 0$ in the category ${}^{\sim}\mathrm{Mod}_S^{\varphi,h}$.

To promote our exact sequence to the category $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, we make induction on n where p^n kills $\mathcal{M}$. The base case $n = 1$ is most complicated and postpone to the end. For general n, by definition, $\mathcal{M}$ sits in the exact sequence in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$: $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ with $\mathcal{M}_1$, $\mathcal{M}_2$ killed by p^{n-1} and p respectively. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_1^{\mathrm{m}} \arrow[r] \arrow[d] & \mathcal{M}_1 \arrow[r] \arrow[d] & \mathcal{M}_1^{\mathrm{n}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}^{\mathrm{m}} \arrow[r] \arrow[d, "f"] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{n}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_2^{\mathrm{m}} \arrow[r] & \mathcal{M}_2 \arrow[r] & \mathcal{M}_2^{\mathrm{n}} \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

We need to show that the first columns is short exact. Note that $\mathcal{M}_2$ is finite S_1-free, the exact sequence in the second column yields the exact sequence $0 \to M_1 \to M \to M_2 \to 0$ where $M_i := \mathcal{M}_i/I_+\mathcal{M}_i$ for $i = 1, 2$. So the sequence $0 \to \mathcal{M}_1^{\mathrm{m}}/I_+ \to \mathcal{M}^{\mathrm{m}}/I_+ \to \mathcal{M}_2^{\mathrm{m}}/I_+ \to 0$ is also exact as it is the same as the exact sequence $0 \to M_1^{\mathrm{m}} \to M^{\mathrm{m}} \to M_2^{\mathrm{m}} \to 0$. Note that $\mathcal{M}_i^{\mathrm{m}}$ is finite S-generated as they are generated by $[M_i^{\mathrm{m}}]$. Note that S_n is coherent ring, see [LL20, Lemma 7.15]. Induction on n and [Sta21, Tag 05CW], we see that $\mathcal{M}$ is coherent and then $\mathcal{M}^{\mathrm{m}}$ is coherent. Since $\mathcal{M}_1^{\mathrm{m}}$ is coherent by induction, $\mathcal{L} = \mathcal{M}^{\mathrm{m}}/\mathcal{M}_1^{\mathrm{m}}$ is also coherent by [Sta21, Tag 05CW] again. Note f induces a map $f' : \mathcal{L} \to \mathcal{M}_2^{\mathrm{m}}$. We need to show that f' is an isomorphism. Let $L = \mathcal{L}/I_+$. Note that $\overline{f}' := f' \mod I_+ : L \to M_2^{\mathrm{m}}$ is an isomorphism. Nakayama Lemma shows that f' is surjective. Let $\mathcal{K} := \ker(f')$ which is still coherent. Since $\mathcal{M}_2^{\mathrm{m}}$ is finite S_1-free by induction, $\mathrm{Tor}_1^S(\mathcal{M}_2^{\mathrm{m}}, S/I_+) = 0$. So we obtain an exact sequence $0 \to \mathcal{K}/I_+ \to L \to M_2^{\mathrm{m}} \to 0$. Hence $\mathcal{K}/I_+ = 0$ as $\overline{f}'$ is an isomorphism. By Nakayama lemma, $\mathcal{K} = 0$ and first column is exact as finite S-module. Using that $\mathcal{M}_2^{\mathrm{m}}$ is finite S_1-free, we see that the sequence $0 \to \mathcal{M}_1^{\mathrm{m}}/\mathrm{Fil}^h S \to \mathcal{M}^{\mathrm{m}}/\mathrm{Fil}^h S \to \mathcal{M}_2^{\mathrm{m}}/\mathrm{Fil}^h S \to 0$ is exact. So the sequence $0 \to \mathrm{Fil}^h S \cdot \mathcal{M}_1^{\mathrm{m}} \to \mathrm{Fil}^h S \cdot \mathcal{M}^{\mathrm{m}} \to \mathrm{Fil}^h S \cdot \mathcal{M}_2^{\mathrm{m}} \to 0$ is exact. Therefore, the first column of (2.11) is exact in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Then it is standard to check that last column is also exact sequence in ${}'\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. In particular, $\mathcal{M}^{\mathrm{n}}$ is an object in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ by induction on n. Once (2.10) is exact in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Then φ on $\mathcal{M}^{\mathrm{m}}$, $\mathcal{M}$ and $\mathcal{M}^{\mathrm{n}}$ defined from $\varphi(x) = c_1^{-h}\varphi_h(x)$ are compatible with maps in the sequence. Since $\mathcal{M}^{\mathrm{m}}$ is generated by $[M^{\mathrm{m}}]$ and $\mathcal{M}^{\mathrm{n}}/I_+ = M^{\mathrm{n}}$, we see that $\mathcal{M}^{\mathrm{m}}$ is multiplicative and $\mathcal{M}^{\mathrm{n}}$ is nilpotent.

Now we discuss the case $n = 1$. First we have shown that $\mathcal{M}^{\mathrm{m}}$ is finite S-generated as the above. Now the exact sequence $0 \to \mathcal{M}^{\mathrm{m}}/I_+ \to \mathcal{M}/I_+ \to \mathcal{M}^{\mathrm{n}}/I_+ \to 0$ is an exact sequence of k-vector spaces. Pick $m_i \in \mathcal{M}^{\mathrm{m}}$ and $n_j \in \mathcal{M}$ so that $m_i \mod I_+$ and $n_j \mod I_+$ are basis of $\mathcal{M}^{\mathrm{m}}/I_+$ and $\mathcal{M}^{\mathrm{n}}/I_+$ respectively. Using that $\mathcal{M}$ is finite S_1-free. It is easy to show that m_i, n_j forms a basis of $\mathcal{M}$ and then both $\mathcal{M}^{\mathrm{m}}$ and $\mathcal{M}^{\mathrm{n}}$ are finite S_1-free. Now it remains to show that $\mathrm{Fil}^h \mathcal{M} \cap \mathcal{M}^{\mathrm{m}} = \mathrm{Fil}^h S\mathcal{M}^{\mathrm{m}}$ so that $\mathrm{Fil}^h \mathcal{M}^{\mathrm{n}} = \mathrm{Fil}^h \mathcal{M}/\mathrm{Fil}^h \mathcal{M}^{\mathrm{m}}$ is a submodule of $\mathcal{M}^{\mathrm{n}}$. Then it is easy to check that $(\mathcal{M}^{\mathrm{n}}, \mathrm{Fil}^h \mathcal{M}^{\mathrm{n}}, \varphi_h)$ is a object in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and thus the sequence $0 \to \mathcal{M}^{\mathrm{m}} \to \mathcal{M} \to \mathcal{M}^{\mathrm{n}} \to 0$ is in the category $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. To show that $\mathrm{Fil}^h \mathcal{M}^{\mathrm{n}} = \mathrm{Fil}^h \mathcal{M}/\mathrm{Fil}^h \mathcal{M}^{\mathrm{m}}$, consider

$\mathcal{F} := \mathcal{M}^{\mathrm{m}}/\operatorname{Fil}^p S_1\mathcal{M}^{\mathrm{m}}$. Write $\widetilde{\operatorname{Fil}}^h\mathcal{F} := (\operatorname{Fil}^h\mathcal{M}\cap\mathcal{M}^{\mathrm{m}})/\operatorname{Fil}^p S_1$ and $\operatorname{Fil}^h\mathcal{F} = \operatorname{Fil}^h S\mathcal{M}^{\mathrm{m}}/\operatorname{Fil}^p S_1$. Since $\operatorname{Fil}^h\mathcal{F} = u^{eh}\mathcal{F}\subset\widetilde{\operatorname{Fil}}^h\mathcal{F}\subset\mathcal{F}$ which is a finite free $k[\![u]\!]/u^{pe}$-module. There exists a basis $e_1,\dots,e_d$ of $\mathcal{F}$ so that $\widetilde{\operatorname{Fil}}^h\mathcal{F}$ is generated by $u^{a_i}e_i$ with $0\le a_i\le eh$. Suppose one of $a_i<eh$. Say $a_1<eh$. Let $\hat e_i\in\mathcal{M}^{\mathrm{m}}$ be a basis which lift e_i. Then $u^{a_1}\hat e_1\in\operatorname{Fil}^h\mathcal{M}\cap\mathcal{M}^{\mathrm{m}}$. So $\varphi_h(u^{eh}\hat e_1) = \varphi_h(u^{eh-a_1}u_i^a\hat e_1) = \varphi(u^{eh-a_1})\varphi_h(u^{a_1}\hat e_1)\in I_+\mathcal{M}$. This contradicts to that $\varphi_h(u^{eh}\hat e_i) \mod I_+$ is a basis $M^m\subset M = \mathcal{M}/I_+$. So all $a_i = eh$ and we have $\operatorname{Fil}^h\mathcal{M}^{\mathrm{m}} = \operatorname{Fil}^h S\cdot\mathcal{M}^{\mathrm{m}} = \operatorname{Fil}^h\mathcal{M}\cap\mathcal{M}^{\mathrm{m}}$ as required. □

Corollary 2.12. *The exact sequence* (2.10) *is canonical in the sense of the following: Suppose $\mathcal{M}$ admits another exact sequence in* $\operatorname{Mod}_{S,\mathrm{tor}}^{\varphi,h}$*:*

$$0\to\widetilde{\mathcal{M}}^{\mathrm{m}}\to\mathcal{M}\to\widetilde{\mathcal{M}}^{\mathrm{n}}\to 0$$

with $\widetilde{\mathcal{M}}^{\mathrm{m}}$ being multiplicative and $\widetilde{\mathcal{M}}^{\mathrm{n}}$ being nilpotent. Then $\widetilde{\mathcal{M}}^{\mathrm{m}} = \mathcal{M}^{\mathrm{m}}$ and $\widetilde{\mathcal{M}}^{\mathrm{n}} = \mathcal{M}^{\mathrm{n}}$.

Proof. Since $\widetilde{\mathcal{M}}^{\mathrm{n}}$ is successive extension of finite free S_1-modules, $\operatorname{Tor}^1_S(\mathcal{M}^{\mathrm{n}}, S/I_+) = 0$. Hence the sequence $0\to\widetilde{\mathcal{M}}^{\mathrm{m}}/I_+\to\mathcal{M}/I_+\to\widetilde{\mathcal{M}}^{\mathrm{n}}/I_+\to 0$ is exact. Since $\widetilde{\mathcal{M}}^{\mathrm{m}}$ is multiplicative, $\widetilde{\mathcal{M}}^{\mathrm{m}}/I_+\subset M^{\mathrm{m}}$ and thus $\widetilde{\mathcal{M}}^{\mathrm{m}}/I_+ = M^{\mathrm{m}}$ otherwise φ on $\widetilde{\mathcal{M}}^{\mathrm{n}}/I_+$ can not be nilpotent. So $[M^{\mathrm{m}}]\subset\widetilde{\mathcal{M}}^{\mathrm{m}}$. Hence $\mathcal{M}^{\mathrm{m}}\subset\widetilde{\mathcal{M}}^{\mathrm{m}}$ as $\mathcal{M}^{\mathrm{m}}$ is constructed as S-submodule of $\mathcal{M}$ generated by $[M^{\mathrm{m}}]$. Since $\mathcal{M}^{\mathrm{m}}/I_+ = M^{\mathrm{m}}$, we have $\widetilde{\mathcal{M}}^{\mathrm{m}} = \mathcal{M}^{\mathrm{m}}$ by Nakayama's lemma. By the definition of exact sequence in the category $\operatorname{Mod}_{S,\mathrm{tor}}^{\varphi,h}$, we see that

$$\operatorname{Fil}^h\widetilde{\mathcal{M}}^{\mathrm{m}} = \widetilde{\mathcal{M}}^{\mathrm{m}}\cap\operatorname{Fil}^h\mathcal{M} = \mathcal{M}^{\mathrm{m}}\cap\operatorname{Fil}^h\mathcal{M} = \operatorname{Fil}^h\mathcal{M}^{\mathrm{m}},$$

where the last equality was proved by the end of the proof of Lemma 2.9. Therefore we have the desired equality $(\widetilde{\mathcal{M}}^{\mathrm{m}}, \operatorname{Fil}^h\widetilde{\mathcal{M}}^{\mathrm{m}}, \varphi_h) = (\mathcal{M}^{\mathrm{m}}, \operatorname{Fil}^h\mathcal{M}^{\mathrm{m}}, \varphi_h)$ as sub-object of $\mathcal{M}$. □

2.3. **Fontaine–Laffaille modules.** Fix $h = p-1$ for this subsection. Let us review Fontaine–Laffaille theory from [FL82]. Let $\mathrm{FM}_{W(k)}$ denote the category whose objects are finite $W(k)$-modules M together with decreasing filtration $\{\operatorname{Fil}^i M\}_{i\ge 0}$ and Frobenius semi-linear map $\varphi_i : \operatorname{Fil}^i M\to M$ satisfying:

1. $\operatorname{Fil}^{i+1} M$ is a direct summand of $\operatorname{Fil}^i M$ for all $i\in\mathbb{N}$, and $\operatorname{Fil}^0 M = M$, $\operatorname{Fil}^{h+1} M = \{0\}$;[3]
2. $\varphi_i|_{\operatorname{Fil}^{i+1} M} = p\cdot\varphi_{i+1}$;
3. $\sum_{i\ge 0}\varphi_i(\operatorname{Fil}^i M) = M$.

Morphisms in $\mathrm{FM}_{W(k)}$ are $W(k)$-linear homomorphisms compatible with filtration and φ_i. It turns out that the category $\mathrm{FM}_{W(k)}$ is abelian, see [FL82, Proposition 1.8]); and any morphism is automatically strict with respect to the filtrations, see [FL82, 1.10 (b)]. A sequence $0\to M_1\to M\to M_2\to 0$ in $\mathrm{FM}_{W(k)}$ is *short exact* if the underlying $W(k)$-module is exact[4]. In this case, we call M_2 a quotient of M. An object $M\in\mathrm{FM}_{W(k)}$ is called *multiplicative* if $\operatorname{Fil}^1 M = \{0\}$ and M is called *nilpotent* if does not have multiplicative subobject. Just as in previous sections, we have the following:

Lemma 2.13. *Let* $(M, \operatorname{Fil}^\bullet M, \varphi_\bullet)\in\mathrm{FM}_{W(k)}$.

(1) It is multiplicative (resp. nilpotent) if and only if φ_0 is bijective (resp. nilpotent).

(2) There is a canonical multiplicative-nilpotent exact sequence in $\mathrm{FM}_{W(k),\mathrm{tor}}$*:*

$$0\longrightarrow M^{\mathrm{m}}\longrightarrow M\longrightarrow M^{\mathrm{n}}\longrightarrow 0 \tag{2.14}$$

so that M^{m} is the maximal multiplicative subobject in M and M^{n} is nilpotent.

Proof. (1): the condition (3) of being an object in $\mathrm{FM}_{W(k)}$ in the case of a multiplicative object translates to φ_0 being surjective, which is equivalent to being bijective due to length consideration. Conversely, if φ_0 is bijective, we let $M'\in\mathrm{FM}_{W(k)}$ be defined as: the underlying module is M itself, with $\operatorname{Fil}^0 M' = M\supset\operatorname{Fil}^1 M' = 0$ and φ_0. Then there is an evident morphism $M'\to M$ in $\mathrm{FM}_{W(k)}$, which is necessarily strict with respect to filtrations (see [FL82, 1.10 (b)]), hence $\operatorname{Fil}^1 M = \operatorname{Fil}^1 M' = 0$. The proof for nilpotent object is in end of the proof of (2).

[3]It turns out that this condition follows from the next two conditions, see [Win84, Proposition 1.4.1 (ii)].

[4]Note that by the above result of Fontaine–Laffaille, the sequence of filtrations are forced to be exact as well.

(2): By Fitting lemma, we have $M = M^{\mathrm{m}} \oplus M^{\mathrm{n}}$, only as φ-modules, so that φ_0 on M^{m} is bijective and φ_0 on M^{n} is nilpotent. Let $\mathrm{Fil}^1 M^{\mathrm{m}} = 0$, we get the desired sequence. The fact that the *quotient* M^{n} with the induced filtration is nilpotent follows from (1). By the exact sequence (2.14), M is nilpotent if and only if $M = M^{\mathrm{n}}$, whose φ_0 is nilpotent. $\square$

For any object M in $\mathrm{FM}_{W(k)}$, we can attach a Breuil module $\underline{\mathcal{M}}_{\mathrm{FM}}(M) \in \mathrm{Mod}_{S\,\mathrm{tor}}^{\varphi,h,\nabla}$ in the following ways: Let $\mathcal{M} = \underline{\mathcal{M}}_{\mathrm{FM}}(M) := S \otimes_{W(k)} M$; $\nabla_{\mathcal{M}} = \nabla_S \otimes \mathrm{id}_M$; $\mathrm{Fil}^h \mathcal{M} := \sum\limits_{i=0}^{h} \mathrm{Fil}^i S \otimes_{W(k)} \mathrm{Fil}^{h-i} M$. By definition $\mathrm{Fil}^h \mathcal{M}$ is a submodule of $\mathcal{M}$. We define $\varphi_{h,\mathcal{M}} : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ by $\varphi_{h,\mathcal{M}} := \sum\limits_{i=0}^{h} \left(\varphi_i \mid_{\mathrm{Fil}^i S}\right) \otimes \left(\varphi_{h-i} \mid_{\mathrm{Fil}^{h-i} M}\right)$, this is well-defined because $\mathrm{Fil}^{i+1} M$ is a direct summand of $\mathrm{Fil}^i M$. It is standard to check that $\underline{\mathcal{M}}_{\mathrm{FM}}(M)$ is a Breuil module in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$.

Proposition 2.15. (1) *Let $M \in \mathrm{FM}_{W(k),\mathrm{tor}}$. Then $\underline{\mathcal{M}}_{\mathrm{FM}}((2.14))$ is isomorphic to* (2.10) *with $\mathcal{M} = \underline{\mathcal{M}}(M)$. In particular, $\underline{\mathcal{M}}(M^{\mathrm{m}}) = \underline{\mathcal{M}}(M)^{\mathrm{m}}$.*

(2) *Given an $M \in \mathrm{FM}_{W(k),\mathrm{tor}}$ and suppose that there exists a classical Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ so that $\underline{\mathcal{M}}(\mathfrak{M}) \simeq \underline{\mathcal{M}}_{\mathrm{FM}}(M)$ in the category of $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. Then we have isomorphism $\underline{\mathcal{M}}_{\mathrm{FM}}((2.14)) \simeq \underline{\mathcal{M}}((2.4))$. In particular, $\underline{\mathcal{M}}(\mathfrak{M}^{\mathrm{n}}) = \underline{\mathcal{M}}(\mathfrak{M})^{\mathrm{n}} = \underline{\mathcal{M}}_{\mathrm{FM}}(M^{\mathrm{n}})$.*

Proof. It is easy to check that if $M \in \mathrm{FM}_{W(k),\mathrm{tor}}$ (resp. $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$) is multiplicative or nilpotent then so is $\underline{\mathcal{M}}_{\mathrm{FM}}(M)$ (resp. $\underline{\mathcal{M}}(\mathfrak{M})$). Then the Proposition follows Corollary 2.12. $\square$

For later use, let us prove the following technical lemma which says that one can test an object in $\mathrm{FM}_{W(k)}$ after looking at its "Breuil's counterpart". This is well-known to experts.

Lemma 2.16. *Let $(M, \mathrm{Fil}^\bullet M, \varphi_\bullet)$ be a filtered module with divided Frobenius, namely only assuming the condition (2) in the definition of $\mathrm{FM}_{W(k)}$ is satisfied. Let $\mathcal{M} = \underline{\mathcal{M}}_{\mathrm{FM}}(M) := S \otimes_{W(k)} M$ and $\mathrm{Fil}^h \mathcal{M} := \sum\limits_{i=0}^{h} \mathrm{Fil}^i S \otimes_{W(k)} \mathrm{Fil}^{h-i} M$. Suppose there is a semi-linear map $\varphi_h \colon \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ satisfying*

$$\varphi_h = \sum_{i=0}^{h} \left(\varphi_i \mid_{\mathrm{Fil}^i S}\right) \otimes \left(\varphi_{h-i} \mid_{\mathrm{Fil}^{h-i} M}\right).$$

Then $(M, \mathrm{Fil}^\bullet M, \varphi_\bullet)$ is an object in $\mathrm{FM}_{W(k)}$ if and only if $\varphi_h(\mathrm{Fil}^h \mathcal{M})$ generates $\mathcal{M}$ as an S-module.

Proof. "Only if" part follows from the standard direction of going from Fontaine–Laffaille modules to Breuil modules as discussed above, below we prove the "if" part, which is the only part that will be used later. To that end, we simply observe that $\mathcal{M}/(p, I_+) \cdot \mathcal{M} \cong M/p$. One checks that the induced map $\overline{\varphi_h} \colon \mathrm{Fil}^h \mathcal{M} \to M/p$ has image given by the image of $\sum_{i=0}^h \overline{\varphi_i} \colon \bigoplus_{i=0}^h \mathrm{Fil}^i M \to M/p$. Our condition now implies the reduction map is surjective. Since M is p-adically complete, it follows that the map $\sum_{i=0}^h \varphi_i \colon \bigoplus_{i=0}^h \mathrm{Fil}^i M \to M$ before mod p is also surjective, which is exactly what we need to show. $\square$

2.4. **Relations to Galois representations.** Fix $\pi_n \in \overline{K}$ so that $\underline{\pi} := (\pi_n) \in \mathcal{O}_{\mathbf{C}}^\flat$ and $\pi_0 = \pi$; $K_\infty := \bigcup_{n \geq 0} K(\pi_n)$ and $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. We embed $\mathfrak{S} \to A_{\mathrm{inf}}$ via $u \mapsto [\underline{\pi}]$. As discussed in [LL20, §6.2], for a classical Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$, we can associate Galois representation of G_∞ via $T_{\mathfrak{S}}(\mathfrak{M}) = (\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathcal{O}_{\mathbf{C}}^\flat))^{\varphi=1}$ and $T_{\mathfrak{S}}^h(\mathfrak{M}) = (\mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes A_{\mathrm{inf}})^{\varphi_h=1}$ where $\mathrm{Fil}^h \varphi^*\mathfrak{M} := \{x \in \varphi^*\mathfrak{M} | (1 \otimes \varphi)(x) \in E^h\mathfrak{M}\}$ and $\varphi_h : \mathrm{Fil}^h \varphi^*\mathfrak{M} \to \varphi^*\mathfrak{M}$ is given by $\varphi_h(x) = \frac{(1\otimes\varphi)(x)}{\varphi(a_0^{-1}E)^h}$. Please consult [LL20, §6.2] for more details of $T_{\mathfrak{S}}^h$ and $T_{\mathfrak{S}}$, for example, $T_{\mathfrak{S}}^h(\mathfrak{M}) = T_{\mathfrak{S}}(\mathfrak{M})(h)$ and both $T_{\mathfrak{S}}$ and $T_{\mathfrak{S}}^h$ are exact.

Note that if $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ has an A_{inf}-semi-linear G_K-action which extends the natural G_∞-action and commutes with φ, then $T_{\mathfrak{S}}(\mathfrak{M})$ is a G_K-representations. In particular, this is the case when $\mathfrak{M} = \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}_n)$ modulo u^∞-torsion.

Now given a Breuil module $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$, then as explained around [LL20, Eqn (6.19)], we define $\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}}) := \mathrm{Fil}^h \mathcal{M} \otimes_S A_{\mathrm{crys}}$ then φ_h extends to $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ and define a G_K-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$: for any $\sigma \in G_K$, any $x \otimes a \in A_{\mathrm{crys}} \otimes_S \mathcal{M}$, define

$$\sigma(x \otimes a) = \sum_{i=0}^{\infty} \nabla^i(x) \otimes \gamma_i\left(\sigma([\underline{\pi}]) - [\underline{\pi}]\right)\sigma(a). \tag{2.17}$$

The above G_K-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ extends the G_∞-action, preserves filtration and commutes with φ_h. As in [LL20, §6.3], we define

$$T_S(\mathcal{M}) := (\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}}))^{\varphi_h = 1},$$

which is a $\mathbb{Z}_p[G_K]$-module.

Now given $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ and let $\mathfrak{c} := \prod_{n=1}^{\infty} \varphi^n(\frac{E}{E(0)}) \in S^\times$. As explained in the proof of [LL20, Prop. 6.12], the map $m \mapsto \mathfrak{c}^h(1 \otimes m)$ induces to natural map $\iota : T^h_{\mathfrak{S}}(\mathfrak{M}) \to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$.

Suppose that $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ has G_K-action which extends G_∞-action and commutes with φ, and the natural map $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to \underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ is compatible with G_K-actions on both sides. Then as explained in [LL20, Remark 6.14], the natural map $T_{\mathfrak{S}}(\mathfrak{M})(h) \simeq T^h_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\iota} T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ is compatible with G_K-actions on the both sides. In particular, this will happen (see the proof of Theorem 5.28) when $\mathfrak{M} = \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)$ is an object in $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ and $\underline{\mathcal{M}}(\mathfrak{M})$ is subobject of $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n)$ inside $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$.

Lemma 2.18. *If $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h,\mathrm{c}}$ is nilpotent then the natural map $\iota : T^h_{\mathfrak{S}}(\mathfrak{M}) \to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ is an isomorphism.*

Proof. Write $\mathcal{M} := \underline{\mathcal{M}}(\mathfrak{M})$. Then $\mathcal{M}$ is also nilpotent by Proposition 2.15. When $h \leq p-2$, ι is known to be isomorphism (without assuming nilpotency of $\mathfrak{M}$) by [LL20, Prop.6.12]. So in the following, we assume $h = p-1$.

Since $T^h_{\mathfrak{S}}$ and $\underline{\mathcal{M}}$ exact and T_S is left exact, we can assume that $\mathfrak{M}$ is killed by p so that $\mathfrak{M}$ is finite free $k[\![u]\!]$-module with basis $e_1, \dots, e_d$. Write $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)A$ with $AB = BA = a_0^{-h}u^{eh}I_d$. Let $\widetilde{e}_i := 1 \otimes e_i$ be basis of $\varphi^*\mathfrak{M}$ and S_1-basis of $\mathcal{M}$. Then $\mathrm{Fil}^h \varphi^*\mathfrak{M}$ is generated by $(\alpha_1, \dots, \alpha_d) = (\widetilde{e}_1, \dots, \widetilde{e}_d)B$ and $\mathrm{Fil}^h \mathcal{M}$ is generated by $(\alpha_1, \dots, \alpha_d)$ and $\mathrm{Fil}^p S_1\mathcal{M}$. Note that $\iota(\widetilde{e}_1, \dots, \widetilde{e}_d) = \mathfrak{c}^h(\widetilde{e}_1, \dots, \widetilde{e}_d)$, and any $x \in (\mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}})$ can be written as $x = (\alpha_1, \dots, \alpha_d)X$ with $X \in (\mathcal{O}^\flat_{\mathbf{C}})^d$ and any $y \in \mathrm{Fil}^h \mathcal{M} \otimes_S A_{\mathrm{crys}}$ can be written as $y = \mathfrak{c}^h(\alpha_1, \dots, \alpha_d)Y + \mathfrak{c}^h(\widetilde{e}_1, \dots, \widetilde{e}_d)Z$ with $Y \in (\mathcal{O}^\flat_{\mathbf{C}}/u^{ep})^d$, $Z \in (\mathrm{Fil}^p A_{\mathrm{crys},1})^d$. Then ι is the same as the following:

$$\{X \in (\mathcal{O}^\flat_{\mathbf{C}})^d | \varphi(X) = BX\} \longrightarrow \{(Y, Z) | Y \in (\mathcal{O}^\flat_{\mathbf{C}}/u^{ep})^d, Z \in (\mathrm{Fil}^p A_{\mathrm{crys},1})^d, \quad \varphi(Y) + \varphi(A)\varphi_h(Z) = BY + Z\}$$

by sending $X \mapsto (X, 0)$. We must show the above map is bijective. For injectivity, note that $X \in \ker(\iota)$ if and only if $BX \in (u^{pe}\mathcal{O}^\flat_{\mathbf{C}})^d$. Then $a_0^{-h}u^{eh}X = ABX \in (u^{pe}\mathcal{O}^\flat_{\mathbf{C}})^d$. Hence $Y = a_0u^{-e}X \in (\mathcal{O}^\flat_C)^d$. Note that $\varphi(X) = BX$ implies that $A\varphi(X) = a_0^{-h}u^{eh}X$ and then $Y = A\varphi(Y)$. So $Y = A\varphi(A)\cdots\varphi^m(A)\varphi^{m+1}(Y)$. Since $\mathfrak{M}$ is nilpotent, $A\varphi(A)\cdots\varphi^m(A) \to 0$ for $m \to \infty$, we see that $Y = 0$. This proves the injectivity of ι.

To prove the surjectivity of ι, consider the equation $\varphi(Y) + \varphi(A)\varphi_h(Z) - BY = Z$. Note that $A_{\mathrm{crys},1} = (\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})[\{z_i\}_{i\geq 1}]/\{z_i^p, i \geq 1\}$ with z_i the image of $\gamma_{p^i}(E)$ in $A_{\mathrm{crys},1}$. Since $\varphi_h(z_i) = a_0^{p^i}$ or 0, the left side of equation is in $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$, this forces the right side $Z = 0$ and we only have $\varphi(Y) = BY$ inside $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$. So it suffices to show there exists $\widetilde{Y} \in (\mathcal{O}^\flat_C)^d$ so that $\varphi(\widetilde{Y}) = B\widetilde{Y}$ and $B\widetilde{Y} = BY$ inside $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$. To prove the existence of $\widetilde{Y}$, pick any lift $Y_0 \in (\mathcal{O}^\flat_{\mathbf{C}})^d$ of Y. Then $\varphi(Y_0) = BY_0 + u^{pe}W_0$. Since $u^{pe}I_d = BA(a_0u^e)^hI_d$, we have $\varphi(Y_0) = BY_1$ with $Y_1 = Y_0 + u^eAa_0^hW_0$. Then $\varphi(Y_1) = BY_1 + u^{pe}\varphi(A)W_1$ for some $W_1 \in (\mathcal{O}^\flat_{\mathbf{C}})^d$. Continue construct Y_n in this way, we have $\varphi(Y_n) = BY_n + u^{pe}A\varphi(A)\cdots\varphi^n(A)W_n$ for some $W_n \in (\mathcal{O}^\flat_{\mathbf{C}})^d$ and then $Y_{n+1} = Y_n + u^eA\varphi(A)\cdots\varphi^n(A)W_n$. Then Y_n converges to $\widetilde{Y}$ as $A\varphi(A)\cdots\varphi^n(A) \to 0$. Since $\widetilde{Y} = Y_0 + u^eA\widetilde{W}$ for some $\widetilde{W} \in (\mathcal{O}^\flat_C)^d$, we see that $B\widetilde{Y} = BY_0 = BY$ inside $(\mathcal{O}^\flat_{\mathbf{C}}/u^{pe})^d$. □

For $h = p-1$ the following example show that the above lemma will fail without $\mathfrak{M}$ being nilpotent.

Example 2.19. Let $h = p-1$. Consider rank 1-Kisin module $\mathfrak{M} = \mathfrak{S}\cdot e_1$ and $\varphi(e_1) = e_1$. Then $\widetilde{e}_1 = 1\otimes e_1$ is a basis of $\varphi^*\mathfrak{M}$ with $\mathrm{Fil}^h \varphi^*\mathfrak{M} = E^h\varphi^*\mathfrak{M}$. We have $\mathcal{M} = \underline{\mathcal{M}}(\mathfrak{M}) = S\cdot \widetilde{e}_1$ with $\mathrm{Fil}^h \mathcal{M} = \mathrm{Fil}^h S\widetilde{e}_1$ and $\varphi_h(x\widetilde{e}_1) = \varphi_h(x)\widetilde{e}_1, \forall x \in \mathrm{Fil}^h S$. Hence

$$T^h_{\mathfrak{S}}(\mathfrak{M}) = (E^h A_{\mathrm{inf}})^{\varphi_h = 1}\widetilde{e}_1 = \{E^h x \in E^h A_{\mathrm{inf}} | \varphi(x) = a_0^{-h}E^h x\}\widetilde{e}_1 = E^h\mathfrak{t}^h\mathbb{Z}_p\widetilde{e}_1.$$

Here $\mathfrak{t}\in A_{\mathrm{inf}}$ is discussed in Example 3.2.3 in [Liu10] which also shows that $\varphi(\mathfrak{t}) = a_0^{-1}E\mathfrak{t}$ and $t = \mathfrak{c}\varphi(\mathfrak{t})$. On the other hand, $T_S(\mathcal{M}) = (\mathrm{Fil}^h A_{\mathrm{crys}})^{\varphi_h=1}\widetilde{e}_1 = \frac{t^h}{p}\mathbb{Z}_p\widetilde{e}_1$. Tracing the definition of $\iota : T^h_{\mathfrak{S}}(\mathfrak{M}) \to T_S(\mathcal{M})$, we see that $\iota(E^h\mathfrak{t}^h\widetilde{e}_1) = t^h\mathbb{Z}_p\widetilde{e}_1 \subset T_S(\mathcal{M}) = \frac{t^h}{p}\mathbb{Z}_p\widetilde{e}_1$. So ι is not a surjection in this case. By modulo p^n, we see $\ker(\iota) \cong \mathrm{coker}(\iota)$ is unramified and killed by p.

Corollary 2.20. *Let $h = p-1$ and $\mathfrak{M} \in \mathrm{Mod}^{\varphi,h,\mathrm{c}}_{\mathfrak{S},\mathrm{tor}}$ be a classical Kisin module of height $p-1$. Then the kernel and cokernel of $\iota : T^h_{\mathfrak{S}}(\mathfrak{M}) \to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ are canonically isomorphic and are unramified representations killed by p.*

Proof. Note that $T^h_{\mathfrak{S}}$ is exact, see [LL20, §6.2]. Since T_S is clearly left exact, by the exact sequence (2.4) and Lemma 2.18, it suffices to prove Corollary for $\mathfrak{M}$ be multiplicative. Clearly, we can assume $k = \overline{k}$ and then $\mathfrak{M}$ is direct sum of the $\mathfrak{S}_n\cdot e_1$ with $\varphi(e_1) = e_1$. Now our desired conclusion just follows from the above Example. □

Finally, given a Fontaine–Laffaille module $M \in \mathrm{FM}_{W(k)}$. Set

$$T_{\mathrm{FM}}(M) := T_S(\underline{\mathcal{M}}_{\mathrm{FM}}(M)) = \mathrm{Fil}^h(M\otimes_{W(k)} A_{\mathrm{crys}})^{\varphi_h=1}$$

where $\mathrm{Fil}^h(M\otimes_{W(k)} A_{\mathrm{crys}}) = \sum\limits_{i=0}^{h} \mathrm{Fil}^i M \otimes_{W(k)} \mathrm{Fil}^{h-i} A_{\mathrm{crys}}$.

3. Boundary degree prismatic cohomology

3.1. **Structure of u^∞-torsion.** Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathcal{O}_K$ which is a degree e totally ramified extension of $W = W(k)$, the Witt ring of a perfect field k of characteristic $p>0$. Let $\mathfrak{M}^i_n$ denote $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$, where $n = \infty$ shall be understood as not modulo any power of p at all.

Proposition 3.1. *We have the following restriction on the annihilator ideal of $\mathfrak{M}^i_n$:*

$$E^{i-1}\cdot \mathrm{Ann}(\mathfrak{M}^i_n) \subset \mathrm{Ann}(\varphi^*\mathfrak{M}^i_n) = \mathrm{Ann}(\mathfrak{M}^i_n)\otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}}\mathfrak{S}.$$

Proof. The equality follows from the flatness of $\varphi_{\mathfrak{S}}$. To show the inclusion, we first observe that

$$\varphi^*\mathfrak{M}^i_n = \left(\varphi^*\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)\right)[u^\infty].$$

Indeed, this follows from the fact that $\varphi_{\mathfrak{S}}$ is flat and sends u to u^p.

To finish the proof, it suffices to show that multiplication by E^{i-1} on $\varphi^*\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)$ factors through a submodule of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)$, as then the same thing is true after replacing the above modules with their u^∞-torsion submodules. Now let us stare at the following diagram

$$(\boxdot)\qquad \begin{tikzcd} \varphi^*\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)\otimes_{\mathfrak{S}}(E^{i-1}) \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \arrow[r] \arrow[d, hook, "\varphi_{i-1}"'] & \varphi^*\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \\ & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n). & \end{tikzcd}$$

Here the top row is given by (mod p^n of) the following inclusions of quasi-syntomic sheaves

$$\Delta^{(1)}\otimes_{\mathfrak{S}}(E^{i-1}) \subset \mathrm{Fil}^{i-1}_{\mathrm{N}} \subset \Delta^{(1)},$$

and φ_{i-1} is the divided Frobenius. Finally apply [LL20, Lemma 7.8.(3)], we see that φ_{i-1} is injective in degree i. □

Corollary 3.2. *Let $\alpha \in \mathbb{Z}_{\geq 0}$ satisfy $\mathrm{Ann}(\mathfrak{M}^i_n) + (p) = (u^\alpha, p)$, then we have*
$$\alpha \leq \frac{e(i-1)}{p-1}.$$

Proof. Using Proposition 3.1, after modulo (p), we get the inclusion
$$E^{i-1} \cdot (u^\alpha) \subset \varphi^*(u^\alpha) = (u^{p\alpha})$$
in $\mathfrak{S}/p = k[\![u]\!]$. Since $E \equiv u^e$ modulo p, the above inclusion translates to the inequality
$$p\alpha \leq e(i-1) + \alpha$$
which is exactly what we need to show. □

Later on we shall exhibit examples showing that the above bound is sharp, see Remark 6.13 (1). Now let us conclude our current knowledge on the u^∞-torsion submodules in prismatic cohomology.

Theorem 3.3. *Recall $\mathfrak{M}^i_n := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$.*

(1) If $e \cdot (i-1) < p-1$, then $\mathfrak{M}^i_n = 0$.
(2) If $e \cdot (i-1) < 2(p-1)$, then $\mathrm{Ann}(\mathfrak{M}^i_n) + (u) \supset (p^{i-1}, u)$.
(3) If $e \cdot (i-1) = p-1$, then $\mathrm{Ann}(\mathfrak{M}^i_n) \supset (p, u)$. Moreover the semi-linear Frobenius on $\mathfrak{M}^i_n$ is bijective. In particular $\mathfrak{M}^i_n$ gives rise to an étale φ-module on k, hence an $\mathbb{F}_p$-representation of G_k or equivalently an unramified $\mathbb{F}_p$-representation of G_K.

Later we shall give an interpretation of the G_k-representation in (3) above, see Theorem 4.14 and Corollary 4.15.

Proof. In the situation of (1), the inequality in Corollary 3.2 gives $\alpha = 0$, hence $\mathrm{Ann}(\mathfrak{M}^i_n) + (p)$ is the unit ideal. Since p is topologically nilpotent, this shows that $\mathrm{Ann}(\mathfrak{M}^i_n)$ is already the unit ideal, hence $\mathfrak{M}^i_n = 0$.

In the situation of (2), the inequality in Corollary 3.2 gives $\alpha < 2$. Therefore we have either $\mathfrak{M}^i_n = 0$ or $\mathrm{Ann}(\mathfrak{M}^i_n) + (p) = (u, p)$. Without loss of generality, we may assume $\mathfrak{M}^i_n \neq 0$ and $\mathrm{Ann}(\mathfrak{M}^i_n) + (p) = (u, p)$. Let us pick an element $f = u + a \in \mathrm{Ann}(\mathfrak{M}^i_n)$ with $a \in p \cdot W(k)$, whose existence is guaranteed by our assumption that $\mathrm{Ann}(\mathfrak{M}^i_n) + (p) = (u, p)$. Let us compute:
$$E^{i-1} \cdot f = (E(u))^{i-1} \cdot (u + a) = (u^{e \cdot (i-1)} + \ldots + a_1 \cdot u + a_0) \cdot (u + a) = \sum_{j=0}^{p-1} \varphi_{\mathfrak{S}}(B_j) \cdot u^j.$$
Proposition 3.1 implies that all of B_i's are in $\mathrm{Ann}(\mathfrak{M}^{\mathrm{i}}_{\mathrm{n}})$. Let us contemplate $C_1 = B_1(0)$: the above equation says $\varphi(C_1) = a_1 \cdot a + a_0$. Since we know $v_p(a_1) \geq i-1$ and $v_p(a_0) = i-1$, we see that $v_p(C_1) = i-1$, which implies $(B_1) + (u) \supset (u, p^{i-1})$.

Lastly we turn to (3). Similarly argued as above, we may assume $\mathfrak{M}^i_n \neq 0$ and $\mathrm{Ann}(\mathfrak{M}^i_n) + (p) = (u, p)$, and our first task is to show $u \in \mathrm{Ann}(\mathfrak{M}^i_n)$. To that end, pick again an element $f = u + a \in \mathrm{Ann}(\mathfrak{M}^i_n)$ with $a \in p \cdot W(k)$. Next we compute
$$E^{i-1} \cdot f = (u^e + p \cdot g_1)^{i-1} \cdot (u + a) = (u^{p-1} + p \cdot g_2) \cdot (u + a) = (u^p + p^{i-1} E(0)^{i-1} \cdot a) \cdot 1 + \sum_{j=1}^{p-1} b_j \cdot u^j$$
By Proposition 3.1, we see that another element of the form $u + b \in \mathrm{Ann}(\mathfrak{M}^i_n)$ with $b \in W(k)$ having a bigger p-adic valuation than that of a. Consequently we have $u \in \mathrm{Ann}(\mathfrak{M}^i_n)$, as $a - b$ and a differ by a unit in $W(k)$. Now we do the trick again:
$$E(u)^{i-1} \cdot u = (u^e + pg(u))^{i-1} \cdot u = u^p + \sum_{j=1}^{i-1} \binom{i-1}{j} u^{1+e(i-1-j)} (pg(u))^j = u^p + \sum_{j=1}^{p-1} B_j u^j$$
with $B_j \in W(k)$. Since $u^p \in \mathrm{Ann}(\varphi^* \mathfrak{M}^i_n)$, we see that $\sum_{j=1}^{p-1} B_j u^j \in \mathrm{Ann}(\varphi^* \mathfrak{M}^i_n)$ and hence each $\varphi^{-1}(B_j)$ is in $\mathrm{Ann}(\mathfrak{M}^i_n)$. From the above expansion, we see that
$$E(u)^{i-1} \cdot u \equiv u^p + (i-1) u^{1+e(i-2)} (pg(u)) \mod p^2.$$

Since $E(u)$ is an Eisenstein polynomial, we see that $g(0)$ is a p-adic unit. This implies that $v_p(B_{1+e(i-2)}) = 1$, so $p \in \mathrm{Ann}(\mathfrak{M}^i_n)$.

Lastly, we need to show the semi-linear Frobenius on $\mathfrak{M}^i_n$ is a bijection. Previous paragraph tells us that $\mathfrak{M}^i_n \simeq k^{\oplus r}$. Let us look at the u^∞-torsion part of diagram $\boxdot$

$$\begin{tikzcd}
\varphi^*\mathfrak{M}^i_n \otimes_{\mathfrak{S}} (E^{i-1}) \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n)[u^\infty] \arrow[r, hook, "\varphi_{i-1}"] \arrow[d] & \mathfrak{M}^i_n \\
E^{i-1} \cdot \varphi^*\mathfrak{M}^i_n \arrow[r, hook] & \varphi^*\mathfrak{M}^i_n. &
\end{tikzcd}$$

We claim the first arrow in the top row is surjective, the middle vertical arrow is injective with image being $E^{i-1} \cdot \varphi^*\mathfrak{M}^i_n$, and the map φ_{i-1} is an isomorphism. We know $\varphi^*\mathfrak{M}^i_n \simeq (k[u]/u^p)^{\oplus r}$, hence $E^{i-1} \cdot \varphi^*\mathfrak{M}^i_n$ is also abstractly isomorphic to $k^{\oplus r}$. Let $\ell(\cdot)$ denote the k-length. The above diagram gives a chain of inequality of lengths

$$r \leq \ell(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n)[u^\infty]) \leq r = \ell(\mathfrak{M}^i_n),$$

where the first inequality follows from previous sentence. So the above inequalities are both equalities, and the claim follows easily. The composition, which we have shown to be surjective, of

$$\mathrm{Frob}_k^*\mathfrak{M}^i_n \xrightarrow{-\otimes_{\mathfrak{S}}(E^{i-1})} \varphi^*\mathfrak{M}^i_n \otimes_{\mathfrak{S}} (E^{i-1}) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n)[u^\infty] \xrightarrow{\varphi_{i-1}} \mathfrak{M}^i_n$$

is the linearization of the semi-linear Frobenius on $\mathfrak{M}^i_n$. This shows that the semi-linear Frobenius on $\mathfrak{M}^i_n$ is surjective, hence bijective by length/dimension considerations. $\square$

Below let us remark on results in literature concerning u^∞-torsion in Breuil–Kisin prismatic cohomology.

Remark 3.4.

(1) Under the assumption $e \cdot i < p - 1$, Min [Min21, Theorem 0.1] showed that the i-th prismatic cohomology has no u-torsion and "looks like" the étale cohomology of the geometric generic fibre. His strategy is to exploit the fact that Frobenius map in degree i has height i. Note that his method also shows that in the same range, the i-th (derived) mod p^n prismatic cohomology also has no u-torsion. But as far as we can tell, the method stops outside the above range.

(2) Philosophically speaking, the u^∞-torsion in i-th (derived) mod p^n prismatic cohomology is surjected on by $(i-1)$-st cohomology of the sheaf Δ_n/u^N for some large N, hence it should secretly have height $(i-1)$. Our Proposition 3.1 may be taken as a manifestation of this philosophy. Later on we show this philosophy is literally true for u^∞-torsion in the integral prismatic cohomology, see Corollary 3.13.

(3) In our previous work, we showed a close relation between u-torsion in prismatic cohomology and structure of Breuil's crystalline cohomology [LL20, Theorem 7.22]. Using this relation, together with Caruso's result [Car08, Theorem 4.1.24], we obtained the same conclusion as in Theorem 3.3.(1) and an improvement of Caruso's result [Car08, Theorem 4.1.24 and Theorem 4.2.1], see [LL20, Corollary 7.25]. Note that our bound on the cohomological index is 1 higher than Caruso's result.

(4) Our control of u-torsion in this paper bypasses Caruso's result. Hence we obtain a proof of Caruso's result and its improvement simultaneously, c.f. [LL20, Theorem 7.22 and Corollary 7.25].

(5) Later on, we shall see that our bound is in some sense sharp by exhibiting an example having (u,p)-torsion with $e = p - 1$ and $i = 2$. See Section 6.

Let us give an application by showing the module structure of prismatic cohomology in low range looks like a $\mathbb{Z}_p$-module.

Corollary 3.5. *Let i be an integer satisfying $e \cdot (i-1) < p - 1$. Then there exists a (non-canonical) isomorphism of $\mathfrak{S}$-modules*

$$\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}.$$

Proof. Since we always have inclusions $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})/p^n \subset \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)$. In the specified range, we know the latter has no u-torsion by Theorem 3.3.(1). Applying Proposition 2.6 shows that there exists an isomorphism of $\mathfrak{S}$-modules

$$\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq N_i \otimes_{\mathbb{Z}_p} \mathfrak{S}$$

for some $\mathbb{Z}_p$-module N_i. To obtain $N_i \simeq \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p)$, we simply use the étale comparison of Bhatt–Scholze, see [BMS18, Theorem 1.8.(iv)] and [BS19, Theorem 1.8.(4)]. Here we are using the fact that the isomorphism class of a finitely generated $\mathbb{Z}_p$-module is determined by its base change to $W(C^\flat)$. □

One should compare with Min's result [Min21, Theorem 5.11]. Our bound on the cohomological degree i here is 1 better than Min's. Below we remind readers a useful result in [BMS18] assuring nice behavior of prismatic cohomology when crystalline cohomology has no torsion, which is a condition often summoned in literature.

Remark 3.6. If $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ is torsion free, then $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ is free. This follows from [BMS18, Corollary 4.17] and crystalline comparison. We sketch a proof below, see also [KP21, Lemma 4.3.28.(1)].

Proof. Let us denote $\mathfrak{M} := \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, we shall use the two short exact sequences appeared in the proof of Proposition 2.6. Crystalline comparison implies $\mathfrak{M}/u\mathfrak{M} \hookrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$. Let us derived modulo the sequence

$$0 \to \mathfrak{M}_{\mathrm{tor}} \to \mathfrak{M} \to \mathfrak{M}_{\mathrm{tf}} \to 0$$

by u. Since $\mathfrak{M}_{\mathrm{tf}}$ is torsion free, we have $\mathfrak{M}_{\mathrm{tor}}/u \hookrightarrow \mathfrak{M}/u$. The target is p-torsion free by assumption whereas $\mathfrak{M}_{\mathrm{tor}}$ consists of p-power torsion, therefore $\mathfrak{M}_{\mathrm{tor}} = 0$ and $\mathfrak{M}_{\mathrm{tf}} = \mathfrak{M}$. Now we again derived modulo the sequence

$$0 \to \mathfrak{M} \to \mathfrak{M}_{\mathrm{fr}} \to \mathfrak{M}_0 \to 0$$

by u to get $\mathfrak{M}_0[u] \hookrightarrow \mathfrak{M}/u$, same argument as above shows $\mathfrak{M}_0[u] = 0$ whereas $\mathfrak{M}_0$ is supported at the maximal ideal of $\mathfrak{S}$. Therefore we again conclude $\mathfrak{M}_0 = 0$ and $\mathfrak{M} = \mathfrak{M}_{\mathrm{fr}}$. □

Let us conclude this subsection by asking some questions.

Question 3.7. Recall $\mathfrak{M}^i_n := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$.

(1) Let β be the smallest exponent such that $p^\beta \in \mathrm{Ann}(\mathfrak{M}^i_n)$, let γ be the exponent such that $\mathrm{Ann}(\mathfrak{M}^i_n) + (u) = (u, p^\gamma)$. Is there a bound on β and γ in terms of e and i?
(2) In light of the example in Section 6, is β and/or γ bounded above by a polynomial in $\log_p$ of a polynomial in e and i, maybe simply bounded above by $\log_p(\frac{e\cdot(i-1)}{p-1}) + 1$ when p is odd?[5]

3.2. **Comparing Frobenius and Verschiebung.** Given a smooth proper formal scheme $\mathcal{X}$ over $\mathcal{O}_K$, for each degree i, we have a natural inclusion $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}/u \hookrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ coming from the crystalline comparison of prismatic cohomology theory. Here the supscript $(-)^{(1)}$ denotes the Frobenius twist, so

$$\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)} := \varphi^*_{\mathfrak{S}}\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}).$$

The map is compatible with Frobenius and Verschiebung, hence induces Frobenius and Verschiebung maps on the cokernel $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$. How to understand these maps? That is the question we shall answer in this subsection.

Given any algebra R which is quasi-syntomic over $\mathcal{O}_K$, we may take its mod π reduction R_0 which is quasi-syntomic over k. This way we obtain a natural map of sites $i\colon k_{\mathrm{qSyn}} \to (\mathcal{O}_K)_{\mathrm{qSyn}}$. Note that the functor $R_0 \mapsto \mathrm{Cris}(R_0/W)$ is a quasi-syntomic sheaf on k_{qSyn}. Here by abuse of notation we use $\mathrm{Cris}(R_0/W)$ to denote the left Kan extended crystalline cohomology. The sheaf $i_*\,\mathrm{Cris}$ takes an algebra R in $(\mathcal{O}_K)_{\mathrm{qSyn}}$ to $i_*\,\mathrm{Cris}(R) := \mathrm{Cris}(R_0/W)$. The base change property and the crystalline comparison of prismatic cohomology [BS19, Theorem 1.8.(1)&(5)] gives us the following exact triangles of sheaves on $(\mathcal{O}_K)_{\mathrm{qSyn}}$:

$$\Delta \xrightarrow{\cdot u} \Delta \to i_*\,\mathrm{Cris}^{(-1)}$$

and

$$\Delta^{(1)} \xrightarrow{\cdot u} \Delta^{(1)} \to i_*\,\mathrm{Cris}$$

where $i_*\,\mathrm{Cris}^{(-1)}(R) = \mathrm{Cris}(R_0/W) \otimes_{W,\varphi^{-1}} W$ is the Frobenius inverse twist.

[5] After contemplating with the image of Whitehead's J-homomorphism, we suspect the above bound should be up by 1 when $p = 2$ and $e \cdot (i-1) \geq 2$.

Proposition 3.8.

(1) *The linear Frobenius maps* $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)} \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ *and* $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)^{(-1)}$ *induces a linear map* $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u] \to \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})[u]$ *which agrees with the linear Frobenius* $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)} \to \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$ *restricted to u-torsion.*

(2) *The semi-linear Frobenius maps* $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)} \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}$ *and* $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ *induces a semi-linear map* $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u] \to \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$. *This map is u^{p-1} times the semi-linear Frobenius on* $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}$ *restricted to u-torsion.*

Note that semi-linearity means u-torsion are only sent to u^p-torsion under the semi-linear Frobenius, after multiplying u^{p-1} we land in u-torsion again.

Proof. Below we use $\mathrm{lin}-\mathrm{Frob}$ (resp. $\mathrm{sl}-\mathrm{Frob}$) to denote the linearized Frobenius (resp. semi-linear Frobenius) on $\Delta^{(1)}$.

(1): this follows from the following commutative diagram

$$\begin{tikzcd}
\Delta^{(1)} \arrow[r, "\cdot u"] \arrow[d, "\mathrm{lin}-\mathrm{Frob}"'] & \Delta^{(1)} \arrow[r] \arrow[d, "\mathrm{lin}-\mathrm{Frob}"'] & i_* \mathrm{Cris} \arrow[d, "i_*(\mathrm{lin}-\mathrm{Frob})"] \\
\Delta \arrow[r, "\cdot u"] & \Delta \arrow[r] & i_* \mathrm{Cris}^{(-1)} .
\end{tikzcd}$$

(2): this follows from the following analogous commutative diagram

$$\begin{tikzcd}
\Delta^{(1)} \arrow[r, "\cdot u"] \arrow[d, "u^{p-1}\cdot(\mathrm{sl}-\mathrm{Frob})"'] & \Delta^{(1)} \arrow[r] \arrow[d, "\mathrm{sl}-\mathrm{Frob}"'] & i_* \mathrm{Cris} \arrow[d, "i_*(\mathrm{sl}-\mathrm{Frob})"] \\
\Delta^{(1)} \arrow[r, "\cdot u"] & \Delta^{(1)} \arrow[r] & i_* \mathrm{Cris} .
\end{tikzcd}$$

□

Remark 3.9. Comparing the above two formulas, the appearance of extra u^{p-1} factor has a natural explanation. Let M be an $\mathfrak{S}$-module, then by flatness of $\varphi_{\mathfrak{S}}$ we know $\big(M \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S}\big)[u] \cong \big(M[u] \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S}\big)[u]$. We may expand the right hand side as $\big(M[u] \otimes_{W,\varphi_W} W\big) \otimes_W (\mathfrak{S}/u^p[u])$. Under this identification, one checks that there is a semi-linear bijection: $M[u] \xrightarrow{\simeq} \big(M[u] \otimes_{W,\varphi_W} W\big) \otimes_W (\mathfrak{S}/u^p[u])$ given by $m \mapsto (m \otimes 1) \otimes u^{p-1}$. Applying this to $M = \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$ gives the relation between (1) and (2) above.

Next we turn to understand the map on $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ induced from Verschiebung maps. We need the following fact on Nygaard filtration.

Lemma 3.10. *The divided Frobenius* $\varphi_{i-1}\colon \mathrm{Fil}^{i-1}_{\mathrm{N}} \to \Delta$ *induces an isomorphism*

$$\varphi_{i-1}\colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}})_{\mathrm{tors}} \xrightarrow{\cong} \mathrm{H}^i_{\Delta}(\mathcal{X})_{\mathrm{tors}}.$$

Proof. Note that we have a commutative diagram of quasisyntomic sheaves:

$$\begin{tikzcd}
\mathrm{Fil}^{i-1}_{\mathrm{N}} \otimes_{\mathfrak{S}} (E) \arrow[rr, "\mathrm{incl}"] \arrow[dr, "\varphi_{i-1}"'] & & \mathrm{Fil}^i_{\mathrm{N}} \arrow[dl, "\varphi_i"] \\
& \Delta &
\end{tikzcd}$$

By [LL20, Lemma 7.8.(3)] we know the i-th divided Frobenius map in degree i is an isomorphism for any bounded prism. Therefore we only need to show the map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}) \otimes_{\mathfrak{S}} (E) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}})$ induces an isomorphism on the torsion submodule.

We claim these modules have the property that their torsion submodule coincides with p^∞-torsion submodule. To see this, just use the fact that both φ_{i-1} and φ_i are injective in degree i, thanks to [LL20, Lemma 7.8.(3)]. The torsion submodule in prismatic cohomology is well-known to coincide with p^∞-torsion submodule.

Therefore we are reduced to showing the above map induces an isomorphism on p^∞-torsion submodule. To that end, we use the exact sequence of quasisyntomic sheaves: $\mathrm{Fil}^{i-1}_{\mathrm{N}} \otimes_{\mathfrak{S}} (E) \to \mathrm{Fil}^i_{\mathrm{N}} \to \mathrm{Fil}^i_{\mathrm{H}}$. Lastly just note that $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}}) \cong \mathrm{H}^0(\mathcal{X}, \Omega^i_{\mathcal{X}/\mathcal{O}_K})$ is p-torsion free. □

Corollary 3.11. *If $e \cdot (i-1) = p-1$, then the map* $\mathrm{incl}\colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}})[u] \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ *is an isomorphism.*

Proof. We stare at the following diagram and contemplate taking H^i:

$$\begin{tikzcd} & {\Delta}^{(1)} \otimes (E)^{\otimes i-1} \arrow[dl, "\varphi\otimes id"'] \arrow[d, "\mathrm{incl}"] \arrow[dr, "\mathrm{incl}"] & \\ {\Delta} \otimes (E)^{\otimes i-1} & \mathrm{Fil}^{i-1}_{\mathrm{N}} \arrow[l, "\varphi"'] \arrow[r, "\mathrm{incl}"] & {\Delta}^{(1)}. \end{tikzcd}$$

By Theorem 3.3 (3), applied to $n = \infty$, combined with Lemma 3.10 we know that the map

$$\mathrm{incl}\colon \left(\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u^\infty]\right)/u \otimes (E)^{\otimes i-1} \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}})[u]$$

is an isomorphism. Using Theorem 3.3 (3) again we know the map

$$\mathrm{incl}\colon \left(\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u^\infty]\right)/u \otimes (E)^{\otimes i-1} \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$$

is also an isomorphism. Therefore we get the desired result. □

The relevance of Nygaard filtration when discussing the Verschiebung map follows from [LL20, Corollary 7.9]. We recall its statement below:

Lemma 3.12. *Let (A, I) be a bounded prism, and let $\mathcal{X}$ be a smooth formal scheme over* $\mathrm{Spf}(A/I)$*. The i-th Verschiebung map (see* [BS19, Corollary 15.5]*)*

$$V_i\colon \tau^{\leq i} {\Delta}_{\mathcal{X}/A} \otimes_A I^{\otimes i} \to \tau^{\leq i} {\Delta}^{(1)}_{\mathcal{X}/A}$$

can be functorially identified with $\mathrm{incl} \circ \varphi^{-1}$*:*

$$\tau^{\leq i} {\Delta}_{\mathcal{X}/A} \otimes_A I^{\otimes i} \xleftarrow[\cong]{\varphi} \tau^{\leq i}\, \mathrm{Fil}^i_{\mathrm{N}}(\mathcal{X}/A) \xrightarrow{\mathrm{incl}} \tau^{\leq i} {\Delta}^{(1)}_{\mathcal{X}/A}.$$

Proof sketch: This follows from the following commutative diagram:

$$\begin{tikzcd} \tau^{\leq i}\, \mathrm{Fil}^i_{\mathrm{N}}(\mathcal{X}/A) \arrow[r, "\varphi", "\cong"'] \arrow[d, "\mathrm{incl}"'] & \tau^{\leq i} {\Delta}_{\mathcal{X}/A} \otimes_A I^{\otimes i} \arrow[dl, dashed, "V_i"'] \arrow[d] \\ \tau^{\leq i} {\Delta}^{(1)}_{\mathcal{X}/A} \arrow[r, "\varphi"] & \tau^{\leq i} {\Delta}_{\mathcal{X}/A}. \end{tikzcd}$$

Here the top arrow is an isomorphism due to [LL20, Lemma 7.8.(3)], the diagonal map is defined affine locally and follows from the description ${\Delta}^{(1)}_{\mathcal{Y}/A} \cong L\eta_I {\Delta}_{\mathcal{Y}/A} \xrightarrow[\cong]{\varphi} {\Delta}_{\mathcal{Y}/A}$ for any smooth affine $\mathcal{Y}$ over $\mathrm{Spf}(A/I)$ (see [BS19, Theorem 15.3]). □

Consequently we see that the torsion and u^∞-torsion in i-th prismatic cohomology is canonically a (generalized) Kisin module of height $(i-1)$.

Corollary 3.13. *The restriction of the Verschiebung map* $V_i\colon \mathrm{H}^i_{\Delta}(\mathcal{X}) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}^{(1)})$ *to either torsion submodule or u^∞-torsion submodule of the source is canonically divisible by E. The division is given by*

$$\mathrm{H}^i_{\Delta}(\mathcal{X})_{\mathrm{tors}} \xleftarrow[\cong]{\varphi_{i-1}} \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}})_{\mathrm{tors}} \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}^{(1)})_{\mathrm{tors}},$$

which, together with the usual prismatic Frobenius, makes the torsion submodule and u^∞-torsion submodule in $\mathrm{H}^i_{\Delta}(\mathcal{X})$ *a (generalized) Kisin module of height $(i-1)$.*

Proof. This follows from combining Lemma 3.10 and Lemma 3.12. □

We can finally understand the induced "Verschiebung" map:

Corollary 3.14. *The i-th linear Verschiebung maps $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}$ and $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)^{(-1)} \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_0/W)$ induces a linear map $V_i \colon \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \to \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$, which fits into the following diagram:*

$$\begin{tikzcd}
\mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}})[u] \arrow[rr, "\varphi_i", "\cong"'] \arrow[dr, "\mathrm{incl}"'] & & \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \arrow[dl, "V_i"] \\
& \mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]. &
\end{tikzcd}$$

In particular the induced map V_i is identified with $\mathrm{incl} \circ \varphi_i^{-1}$.

In other words, the induced V_i is the restriction of "V_{i+1} divided by E" (from Corollary 3.13) to the u-torsion submodule.

Proof. This follows from combining Lemma 3.10 and Lemma 3.12. ☐

Corollary 3.15. *If $e \cdot (i-1) = p-1$, then the induced Verschiebung $V_{i-1} \colon \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u]$ is an isomorphism.*

Proof. This follows from Corollary 3.14, Lemma 3.10 and Corollary 3.11. ☐

Summary. Let us summarize our knowledge on the structure of prismatic cohomology, with the auxiliary $(i-1)$-st Nygaard filtration in mind. Fix the cohomological degree i and $n \in \mathbb{N} \cup \{\infty\}$. The relevant diagram is:

$$\begin{tikzcd}
\Delta^{(1)} \arrow[r, "\otimes(E^{i-1})"] \arrow[dr, "\varphi"'] & \mathrm{Fil}^{i-1}_{\mathrm{N}} \arrow[r, "\otimes(E)"] \arrow[d, "\varphi_{i-1}"] \arrow[l, dashed, bend right, "\mathrm{incl}"'] & \mathrm{Fil}^i_{\mathrm{N}} \arrow[dl, "\varphi_i"] \arrow[l, dashed, bend right, "\mathrm{incl}"'] \\
& \Delta. &
\end{tikzcd}$$

If we drop dotted arrows, then the diagram commutes. On the other hand, the two circles on top has the property that compose the two arrows either way gives multiplication by E^{i-1} and E separately.

The above diagram induces a diagram:

$$\begin{tikzcd}
\varphi^*_{\mathfrak{S}}\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \arrow[r, "f"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \arrow[r, "g"] \arrow[l, bend right] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \arrow[l, bend right]
\end{tikzcd}$$

Here are some knowledge of the above diagram.

(1) The two arrows in the first circle composes either way gives multiplication by E^{i-1}.
(2) The two arrows in the second circle composes either way gives multiplication by E.
(3) Compose the rightward arrows gives the prismatic Frobenius.
(4) Compose the leftward arrows gives the prismatic Verschiebung V_i, see [BS19, Corollary 15.5], [LL20, Corollary 7.9] and Lemma 3.12.
(5) The map g is injective, see [LL20, Lemma 7.8.(3)].
(6) When $n = \infty$, then g induces an isomorphism of torsion submodules, see Lemma 3.10. Hence as far as torsion or u^∞-torsion in $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ is concerned, we may focus on the first circle and see that these Frobenius modules are canonically (generalized) Kisin modules of height $(i-1)$, see Corollary 3.13.

3.3. **Induced Nygaard filtration.** Lastly let us discuss the induced Nygaard filtration on u^∞-torsion in the boundary degree prismatic cohomology.

Lemma 3.16. *Assume $e \cdot (i-1) = p-1$ and let $n \in \mathbb{N} \cup \{\infty\}$. For any $j \in \mathbb{N}$, consider the induced map on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, -/p^n)$ of the maps of quasi-syntomic sheaves $\mathrm{Fil}^j_{\mathrm{N}} \to \Delta^{(1)}$, the following two submodules of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)[u^\infty]$*

- $\mathrm{Im}\big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)\big) \cap \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty]$; *and*
- $\mathrm{Im}\big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)[u^\infty] \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty]\big)$

agree.

Proof. We look at the following diagram of $\mathfrak{S}$-modules, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)[u^\infty] \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \arrow[r] \arrow[d, "f"] & Q_1 \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty] \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n) \arrow[r] & Q_2 \arrow[r] & 0
\end{tikzcd}$$

By [LL20, Proposition 7.12], the $\mathrm{Ker}(f)$ has finite length, hence must be contained in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)[u^\infty]$. The snake lemma implies that $\mathrm{Ker}(g)$ embeds inside a quotient of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty]$. Since $\mathrm{Ker}(g)$, being a submodule of Q_1, is u-torsion free, we see it must be zero, which is exactly what we need to show. □

If no confusion would arise, when $e \cdot (i-1) = p-1$, we shall refer to the submodule in the above lemma as the *(induced) j-th Nygaard filtration* on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty]$. The following proposition reveals what this filtration is.

Proposition 3.17. *Assume $e \cdot (i-1) = p-1$ and let $n \in \mathbb{N} \cup \{\infty\}$.*

(1) The Nygaard filtrations on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty]$ as above is the $E(u) \equiv u^e$-adic filtration.
(2) The map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)$ is injective.
(3) For any $j \geq 0$, the map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i+j}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)$ has kernel given by u^∞-torsion of the source.

Remark 3.18.

(1) We remind readers that, under the hypothesis of this lemma, Theorem 3.3 (3) gives a canonical isomorphism of $\mathfrak{S}$-modules
$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty] \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}_n)[u^\infty] \otimes_{\mathfrak{S}, \varphi_{\mathfrak{S}}} \mathfrak{S} \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}_n)[u, p] \otimes_k \mathfrak{S}/(p, u^p).$$
Therefore the $E(u)$-adic filtration is the same as u^e-filtration.
(2) Also note that $u^{e \cdot (i+j)} = u^{p-1+e \cdot (j+1)} \in (u^p)$ if $j \geq 0$, hence (1) implies (3).
(3) To put Proposition 3.17 (3) in context, let us point out [LL20, Corollary 7.9] which says that the divided Frobenius $\varphi_{i+j} \colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i+j}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}_n)$ is an isomorphism for all $j \geq 0$.

Proof of Proposition 3.17. Throughout this proof, all filtrations referred to are filtrations on $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}^{(1)}_n)[u^\infty]$.

Since we have containment of quasi-syntomic sheaves: $E^j \cdot \mathbbl{\Delta}^{(1)} \subset \mathrm{Fil}^j_{\mathrm{N}} \subset \mathbbl{\Delta}^{(1)}$, one easily sees the Nygaard filtration contains the u^e-adic filtration. All we need to show is the converse containment.

Let us first show (1) holds for the $(i-1)$-st Nygaard filtration and (2). As discussed above, since $u^{e \cdot (i-1)} = u^{p-1}$, we see that the $(i-1)$-st Nygaard filtration has length at least that of $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}_n)[u^\infty]$.[6] To finish, it suffices to show that the u^∞-torsion in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n)$ has length at most that. This follows from the fact that the divided Frobenius, which is $\mathfrak{S}$-linear,
$$\varphi_{i-1} \colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i-1}_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbbl{\Delta}_n),$$
is injective, see [LL20, Lemma 7.8.(3)].

Next we show (1) holds for j-th filtration whenever $0 \leq j \leq i-1$. Now we look at another containment of quasi-syntomic sheaves: $E^{i-1-j} \cdot \mathrm{Fil}^j_{\mathrm{N}} \subset \mathrm{Fil}^{i-1}_{\mathrm{N}} \subset \mathrm{Fil}^j_{\mathrm{N}}$. Therefore we see the j-th filtration can differ with the $(i-1)$-st filtration by at most $u^{e \cdot (i-1-j)}$, this gives the desired converse containment by what we proved in the previous paragraph.

Finally we show (1) holds for $(i+j)$-th filtration for any $j \geq 0$, note that this implies (3) as remarked right after the statement of this proposition. We want to show the map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{i+j}_{\mathrm{N}}/p^n)[u^\infty] \to$

[6] Note that here we are not twisting the prismatic cohomology by Frobenius.

$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n^{(1)})[u^\infty]$ is the zero map when $j \geq 0$. Since this map factors through the $j = 0$ case, it suffices to prove the $j = 0$ case. To that end, we shall utilize [LL20, Corollary 7.9]: according to loc. cit. we need to show the prismatic Verschiebung annihilates $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty]$. Now we contemplate the following sequence of arrows:

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n^{(1)})[u^\infty] \xrightarrow{\varphi} \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty] \xrightarrow{\psi} \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n^{(1)})[u^\infty].$$

Here $\psi = V_i$ is the i-th Verschiebung as in [BS19, Corollary 15.5]. The composition of these two arrows is multiplication by $E^i = u^{p-1+e} = 0$, as the module is abstractly several copies of $\mathfrak{S}/(p, u^p)$. Finally we finish the proof by recalling Theorem 3.3 (3) that the map φ above is surjective. $\square$

As a consequence, in the boundary degree, we can use torsion in cohomology of $\mathcal{O}_{\mathcal{X}}$ to bound u^∞-torsion.

Corollary 3.19. *Assume $e \cdot (i-1) = p-1$ and let $n \in \mathbb{N} \cup \{\infty\}$. The natural map $\mathbb{\Delta}^{(1)} \to \mathrm{gr}^0_{\mathrm{N}} \mathbb{\Delta}^{(1)} \cong \mathcal{O}$ gives rise to a canonical injection:*

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty] \otimes_k \big(\mathcal{O}_K/p\big) \hookrightarrow \mathrm{H}^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/p^n).$$

Proof. The exact sequence $\mathrm{Fil}^1_{\mathrm{N}} \to \mathbb{\Delta}^{(1)} \to \mathcal{O}_X$ tells us that the kernel of the map

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n^{(1)})[u^\infty] \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty] \otimes_k \overline{k}[u]/(u^p) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n)$$

is given by the induced first Nygaard filtration on the source, which we know is exactly u^e times the source, thanks to Proposition 3.17 (1). Notice that, as an $\mathcal{O}_K$-algebra, we have $\overline{k}[u]/(u^e) \cong \mathcal{O}_K/p$. Therefore we get the desired injection

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty] \otimes_k \overline{k}[u]/(u^e) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty] \otimes_k \big(\mathcal{O}_K/p\big) \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

$\square$

4. Geometric applications

4.1. **The discrepancy of Albanese varieties.** In this subsection, we give a geometric interpretation of u-torsion in the second Breuil–Kisin prismatic cohomology. Our main application in this subsection has partly been obtained by Raynaud in [Ray79], our method is of course quite different. Without loss of generality, we assume our smooth proper (formal) scheme $\mathcal{X}$ has an $\mathcal{O}_K$-point. This can be arranged after an unramified extension of $\mathcal{O}_K$.

The generic fibre of $\mathcal{X}$ is a smooth proper rigid space X over $\mathrm{Sp}(K)$ admitting a K-point. Specialize the main result of [HL00] to our case where X has a smooth proper formal model, we know the $\mathrm{Pic}^0(X)$ is an abeloid variety which has good reduction, namely it is the rigid generic fibre of a formal abelian scheme over $\mathcal{O}_K$. In the algebraic situation, the existence of abelian scheme integral model follows from Serre–Tate's generalization [ST68] of the Néron–Ogg–Shafarevich's criterion. For the general theory of Néron model of abeloid variety, we refer readers to [Lüt95]. Now we can form the Albanese of X, which is a universal map

$$g_K \colon X \to A$$

from X to abeloid varieties, see [HL20, Section 4]. Since in this case A is the dual of $\mathrm{Pic}^0(X)$, we know it also has good reduction: namely the Néron model of A is a formal abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$. Lastly since $\mathcal{X}$ is smooth over $\mathcal{O}_K$, the Néron mapping property implies that the map $X \to A$ extends uniquely to

$$g \colon \mathcal{X} \to \mathcal{A}$$

over $\mathcal{O}_K$. Take the special fibre of the above map, we get

$$g_0 \colon \mathcal{X}_0 \to \mathcal{A}_0.$$

Now the Albanese theory tells us the above map factors:

$$\begin{tikzcd} \mathcal{X}_0 \arrow[rr, "g_0"] \arrow[dr, "h"'] & & \mathcal{A}_0 \\ & \mathrm{Alb}(\mathcal{X}_0) \arrow[ur, "f"'] & \end{tikzcd}$$

where $\mathrm{Alb}(\mathcal{X}_0)$ is the Albanese of $\mathcal{X}_0$. Therefore, out of a pointed smooth proper formal scheme $\mathcal{X}$ over $\mathcal{O}_K$, we can cook up a map $f\colon \mathrm{Alb}(\mathcal{X}_0) \to \mathcal{A}_0$ of abelian varieties over k. What can we say about this map?

Proposition 4.1. *The map $f\colon \mathrm{Alb}(\mathcal{X}_0) \to \mathcal{A}_0$ above is an isogeny of p-power degree.*

Proof. It suffices to show that f induces an isomorphism of the first ℓ-adic étale cohomology for all primes $\ell \neq p$. From now on we fix such an ℓ. Usual Albanese theory tells us that the Albanese maps $h\colon \mathcal{X}_0 \to \mathrm{Alb}(\mathcal{X}_0)$ and $g_K\colon X \to A$ induces an isomorphism of the first ℓ-adic étale cohomology. To finish the proof, we just use the smooth and proper base change theorems in étale cohomology theory to see that the map g_0, being reduction of the "smooth proper model" g of g_K, also induces an isomorphism of the first ℓ-adic étale cohomology. Since $h^* \circ f^* = g_0^*$ and both h^* and g_0^* induces an isomorphism of the first ℓ-adic étale cohomology, we conclude that f^* also does. □

Let us denote the finite p-power order group scheme $\ker(f)$ by G. The Dieudonné module of G is related to $\mathcal{X}$ in the following way.

Theorem 4.2. *We have an isomorphism of W-modules*

$$\mathbb{D}(G) \cong \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u].$$

Under this identification, the semi-linear Frobenius F on the left hand side and the semi-linear Frobenius φ on the left hand side are related via $F = u^{p-1} \cdot \varphi$, and the linear Verschiebung on the left hand side can be understood as

$$\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \xleftarrow[\cong]{\varphi_1} \mathrm{H}^2_{\mathrm{qSyn}} \mathrm{Fil}^1_{\mathrm{N}}(\mathcal{X}/\mathfrak{S})[u] \xrightarrow{incl} \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}[u].$$

Proof. The Dieudonné module of G in our situation is given by

$$\mathbb{D}(G) \cong \mathrm{Coker}\left(f^*\colon \mathrm{H}^1_{\mathrm{crys}}(\mathcal{A}_0/W) \to \mathrm{H}^1_{\mathrm{crys}}(\mathrm{Alb}(\mathcal{X}_0)/W)\right),$$

so we need to understand the above map f^*.

We want to relate everything to $\mathcal{X}$. First by [Ill79, Remarque 3.11.2] we know the map

$$h^*\colon \mathrm{H}^1_{\mathrm{crys}}(\mathrm{Alb}(\mathcal{X}_0)/W) \to \mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}_0/W)$$

is an isomorphism. Therefore by composing with h^* we have

$$\mathbb{D}(G) \cong \mathrm{Coker}\left(g_0^*\colon \mathrm{H}^1_{\mathrm{crys}}(\mathcal{A}_0/W) \to \mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}_0/W)\right).$$

Next we use the crystalline comparison of prismatic cohomology [BS19, Theorem 1.8.(1)], and get the following diagram

$$\begin{tikzcd}
\mathrm{H}^1_{\Delta}(\mathcal{A}/\mathfrak{S})^{(1)} \arrow[r, two heads] \arrow[d, "\cong"', "g^*"] & \mathrm{H}^1_{\Delta}(\mathcal{A}/\mathfrak{S})^{(1)}/u \arrow[r, "\cong"] \arrow[d, "\cong"', "g^*"] & \mathrm{H}^1_{\mathrm{crys}}(\mathcal{A}_0/W) \arrow[d, "g_0^*"] \\
\mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)} \arrow[r, two heads] & \mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}/u \arrow[r, hook] & \mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}_0/W).
\end{tikzcd}$$

We postpone the proof of the left (and therefore the middle) vertical arrow being an isomorphism of φ-modules over $\mathfrak{S}$ to the next Proposition. The right horizontal arrows are injective because of the standard sequence $0 \to \mathrm{H}^i_{\mathrm{qSyn}}(\Delta^{(1)})/u \to \mathrm{H}^i_{\mathrm{qSyn}}(\Delta^{(1)}/u) \to \mathrm{H}^{i+1}_{\mathrm{qSyn}}(\Delta^{(1)})[u] \to 0$. The top right horizontal arrow is an isomorphism, as the (Breuil–Kisin) prismatic cohomology of abelian schemes are finite free, which in turn follows from the torsion-freeness of the crystalline cohomology of abelian varieties and Remark 3.6. The above diagram and sequence tells us that g_0^* is injective with cokernel given by $\mathrm{H}^2_{\Delta^{(1)}}(\mathcal{X}/\mathfrak{S})[u]$. The description of (the semi-linear) Frobenius follows from Proposition 3.8 (2), and the description of the linear Verschiebung follows from Corollary 3.14. □

The following Proposition was summoned in the above proof.

Proposition 4.3.

(1) The underlying $\mathfrak{S}$-module of $\mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S})$ is finite free; and

(2) *The map* $g^*\colon \mathrm{H}^1_{\Delta}(\mathcal{A}/\mathfrak{S}) \to \mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S})$ *is an isomorphism of Kisin modules. Therefore the Frobenius-twisted version* $g^*\colon \mathrm{H}^1_{\Delta}(\mathcal{A}/\mathfrak{S})^{(1)} \to \mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S})^{(1)}$ *is also an isomorphism.*

Proof. (1): this follows from Corollary 3.5. Alternatively we can prove this using Remark 3.6, [Ill79, Remarque 3.11.2] and the fact that crystalline cohomology of abelian varieties are torsion-free.

(2): Since étale realization of finite free Kisin modules is fully faithful, see [Kis06, Proposition 2.1.12] and also [BS21, Theorem 7.2], we are reduced to checking the étale realization of g^* is an isomorphism. Since the map $\mathfrak{S} \to A_{\inf}$ sending u to $[\underline{\pi}]$ is p-completely faithfully flat, it remains so after p-completely inverting u and $[\underline{\pi}]$ respectively. Therefore we are further reduced to proving it for the $W(C^\flat)$-étale realizations. Now the étale comparison [BS19, Theorem 1.8.(4)] translates the above to the statement that g_K induces an isomorphism of first p-adic étale cohomology, which follows from the usual Kummer sequence together with the fact that the Picard variety of X is an abeloid.[7] □

We get two consequences from Theorem 4.2.

Corollary 4.4. *The finite group scheme G is connected.*

Proof. Since the induced Frobenius on $\mathbb{D}(G)$, when identified with $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \subset \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$, is divisible by u^{p-1}, powers of Frobenius will gain more and more u-divisibility. We see the Frobenius is nilpotent as $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u] \subset \mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]$ and there is a power of u which kills the latter. Now Theorem 4.2 implies the Frobenius on $\mathbb{D}(G)$ is nilpotent, therefore G is connected. □

Remark 4.5. The above fact can actually be seen directly. Let us quotient out $\mathrm{Alb}(\mathcal{X}_0)$ by the neutral component subgroup scheme of G, denoted by $\mathcal{A}'_0$. Then we get a factorization $\mathcal{X}_0 \to \mathcal{A}'_0 \xrightarrow{f'_0} \mathcal{A}_0$ of g_0. Now f'_0 is finite étale by construction. Hence deformation theory implies the above sequence lift to $\mathcal{X} \to \mathcal{A}' \xrightarrow{f'} \mathcal{A}$ with $\mathcal{A}'$ being a formal abelian scheme finite étale above $\mathcal{A}$. Now the composition of the above map is the universal map from $\mathcal{X}$ to formal abelian schemes as pointed formal schemes,[8] we conclude that the map f' has to be an isomorphism, hence the neutral component subgroup scheme of G is G itself.

Combining Theorem 4.2, Theorem 3.3 (with $i = 2$) and Corollary 3.15, we immediately yield the following result.

Corollary 4.6.

(1) *If $e < p-1$ then the map $f\colon \mathrm{Alb}(\mathcal{X}_0) \to \mathrm{Alb}(X)_0$ is an isomorphism.*
(2) *If $e < 2(p-1)$ then $\ker(f)$ is p-torsion.*
(3) *If $e = p-1$ then $\ker(f)$ is p-torsion and of multiplicative type, hence must be a form of several copies of μ_p. Moreover there is a canonical injection of $\mathcal{O}_K$-modules $\mathbb{D}(\ker(f)) \otimes_k \left(\mathcal{O}_K/p\right) \hookrightarrow \mathrm{H}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.*

Proof. (1) and (2) follows from Theorem 4.2 and Theorem 3.3 (with $i = 2$) (1) and (2) respectively. As for the multiplicativity claim in (3): recall that a finite flat group scheme over k is of multiplicative type if and only if its Dieudonné module has bijective Verschiebung, hence the claim follows from Theorem 4.2 and Corollary 3.15. The last sentence follows from Corollary 3.19. □

When $e = 1$ and $p = 2$ the above says that although the f need not be an isomorphism, the kernel is always a 2-torsion, such an interesting example can be found in [BMS18, Subsection 2.1], and one can check directly that the example there does satisfy our prediction here. In fact the f for their example can be identified with the relative Frobenius of an ordinary elliptic curve (which is the reduction of the E in their notation) over $\mathbb{F}_2$. For a generalization of this example to the case when $p \neq 2$, we refer readers to our Section 6, and specifically our Remark 6.10 and Remark 6.13 (3).

[7] Note that in general Albanese of smooth proper rigid spaces (granting its existence) always induces an injective but not necessarily surjective map of first étale cohomology, no matter ℓ-adic or p-adic, see [HL20, Proposition 4.4, Example 5.2 and Example 5.8]. The surjectivity is equivalent to the Picard variety being an abeloid (assuming p is invertible in the ground non-archimedean field).

[8] We use the Néron mapping property and the fact that the generic fibre map being the Albanese map.

Remark 4.7. If $\mathrm{Pic}^0(\mathcal{X}_0)$ is reduced, then the relative (formal) Picard scheme of $\mathcal{X}/\mathcal{O}_K$ is a formal abelian scheme which is the Néron model of the Picard variety of X/K. Base change property of relative Picard functor now guarantees that the f we have been studying is an isomorphism in this case. Therefore combining with Theorem 4.2 we see that $\mathcal{X}_0$ having reduced Picard scheme implies the second prismatic cohomology of $\mathcal{X}$ has no u-torsion.

The dual question of what we discussed here was studied by Raynaud [Ray79], below we recall some of the main results in loc. cit. and compare with ours.

Remark 4.8. Using determinant construction [KM76], the universal line bundle on $\mathcal{X}_K \times_K \mathrm{Pic}^0(\mathcal{X}_K)$ extends to a line bundle on $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{P}$, where $\mathcal{P}$ is the formal Néron model of $\mathrm{Pic}^0(\mathcal{X}_K)$ which is itself a formal abelian scheme over $\mathcal{O}_K$. Here we used the regularity of $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{P}$ so that any coherent sheaf on it can be presented as a perfect complex, in order to perform the determinant construction. Moreover if we rigidify using the given point $x \in \mathcal{X}(\mathcal{O}_K)$, then the extension as a rigidified line bundle is unique. Taking the special fibre, we get an induced map $\mathcal{P}_0 \to \mathrm{Pic}^0(\mathcal{X}_0)$ which necessarily factors through the reduced subvariety of the target $f^\vee \colon \mathcal{P}_0 \to \mathrm{Pic}^0(\mathcal{X}_0)_{\mathrm{red}}$. By construction, the map $f^\vee$ is dual to the map f we considered before.

Raynaud has studied the question of whether $f^\vee$ is an isomorphism in [Ray79]. His main result says:

(1) [Ray79, Thèoréme 4.1.3.(2)] When $e < p-1$, then $f^\vee$ is an isomorphism.
(2) [Ray79, Thèoréme 4.1.3.(3)] When $e = p-1$, then $\ker(f^\vee)$ is p-torsion and unramified.

We see that his results are the same as Corollary 4.6 (1) and first half of (3), our slight improvement is Corollary 4.6 (2) and second half of (3): We prove the map $f^\vee$ has p-torsion kernel in a larger range of ramifications, and when $e = p-1$ the second cohomology of structure sheaf needs to have "actual" p-torsion in order for $\ker(f)$ to be nonzero. On the other hand, Raynaud's result allows $\mathcal{X}$ to be singular: for instance he just needs $\mathcal{X}_0$ to be normal. Our method crucially relies on prismatic theory, which seems to only work well with local complete intersection singularities. Whether our Corollary 4.6 can be extended to the generality considered by Raynaud remains unclear and interesting to us.

Remark 4.9. One of the key ingredient letting Raynaud to prove the aforementioned results in [Ray79] is an earlier result of his [Ray74] concerning prolongations of finite flat commutative group schemes. In the end of this paper, Section 6.1, we shall see a way to go backward: applying these structural results on Picard/Albanese varieties to a marvelous construction due to Bhatt–Morrow–Scholze [BMS18], one deduces Raynaud's prolongation theorem.

4.2. **The p-adic specialization maps.** Another reason why one might care about u^∞-torsion is because it appears naturally in understanding the specialization map of p-adic étale cohomology or, phrased differently, the p-adic vanishing cycle.

Let us introduce some notations. Fix a complete algebraically closed non-archimedean extension C of K, with ring of integers $\mathcal{O}_C$. Denote the perfect prism associated with $\mathcal{O}_C$, which is known to be oriented, by $(A_{\mathrm{inf}}, (\xi))$. Given p-adic formal scheme $\mathcal{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$, we denote its base change to $\mathcal{O}_C$ (resp. C) as $\mathcal{X}_{\mathcal{O}_C}$ (resp. $\mathcal{X}_C$). Denote the central fibre of $\mathcal{X}_{\mathcal{O}_C}$ by $\mathcal{X}_{\overline{k}}$. We keep assuming $\mathcal{X}$ to be smooth and proper over $\mathrm{Spf}(\mathcal{O}_K)$.

Recall the proper base change theorem gives, for any prime ℓ, a specialization map [Sta21, Tag 0GJ2]

$$\mathrm{Sp}\colon \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}_\ell) \to \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_\ell).$$

The cone of specialization map above is called the vanishing cycle (of $\mathbb{Z}_\ell$). The smooth base change theorem says that the above map is an isomorphism for any $\ell \neq p$ [Sta21, Tag 0GKD], in other words ℓ-adic vanishing cycle vanishes in our setting. On the other hand, one may ask what happens when $\ell = p$. Fix a cohomological degree i and $n \in \mathbb{N} \cup \{\infty\}$, let us look at

$$\mathrm{Sp}^i_n\colon \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n),$$

when $n = \infty$ the above means $\mathbb{Z}_p$ coefficient and we will simply write Sp^i. It is well-known that Sp^i is almost never surjective unless for trivial reasons such as the target being 0. We shall contemplate with $\ker(\mathrm{Sp}^i)$ in this subsection.

In [BS19, Section 9] one finds a prismatic interpretation of the p-adic specialization map:

Theorem 4.10 ([BS19, Theorem 9.1 and Remark 9.3]). *There are canonical identifications:*

$$\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n) \cong \big((\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}/p^n\big)^{\varphi=1},$$

and

$$\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)^{\varphi=1},$$

fitting in the following diagram, which is commutative up to coherent homotopy:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \arrow[r, "\cong"] \arrow[d, "\mathrm{Sp}"'] & \big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)^{\varphi=1} \arrow[d, "\mathrm{incl}"] \\
\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n) \arrow[r, "\cong"] & \big((\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}/p^n\big)^{\varphi=1}.
\end{tikzcd}$$

Here $(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}$ denotes the p-completion of the localization, which is only relevant in the statement when $n = \infty$. This theorem is true without smooth or proper assumption on $\mathcal{X}$: one may safely replace $\mathcal{X}_{\mathcal{O}_C}$ over $\mathrm{Spf}(\mathcal{O}_C)$ with any p-adic formal scheme $\mathcal{Y}$ over a perfectoid base ring as in loc. cit.

Sketch of proof following that of loc. cit. The first identification is [BS19, Theorem 9.1], the second identification is [BS19, Remark 9.3] with details left to readers, so let us fill in some details.

We follow the proof of [BS19, Theorem 9.1]. First we see that both of $\mathrm{R}\Gamma_{\text{ét}}((-)_{\overline{k}}, \mathbb{Z}/p^n)$ and $\big(\Delta_{-/A_{\mathrm{inf}}}/p^n\big)^{\varphi=1}$ are arc-sheaves (see [BM21] for more details on this notion) on $\mathrm{fSch}_{/\,\mathrm{Spf}(\mathcal{O}_C)}$. The former is [BM21, Theorem 5.4], the latter follows from the same argument as in loc. cit.: using [BS19, Lemma 9.2] one has an identification $\big(\Delta_{-/A_{\mathrm{inf}}}/p^n\big)^{\varphi=1} \cong \big(\Delta_{-/A_{\mathrm{inf}},\mathrm{perf}}/p^n\big)^{\varphi=1}$, then one again uses [BS19, Corollary 8.10] to see the latter is an arc-sheaf.

Since everything involved is an arc-sheaf and is arc-locally supported in cohomological degree 0, the relevant maps (of arc-sheaves) live in mapping spaces with contractible components. Altogether we get the following diagram which commutes up to coherent homotopy:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{ét}}((-)_{\overline{k}}, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\mathrm{Sp}"'] & \big(\Delta_{-/A_{\mathrm{inf}}}/p^n\big)^{\varphi=1} \arrow[d, "\mathrm{incl}"] \\
\mathrm{R}\Gamma_{\text{ét}}((-)_C, \mathbb{Z}/p^n) \arrow[r, "\cong"] & \big((\Delta_{-/A_{\mathrm{inf}}}[1/\xi])^{\wedge}/p^n\big)^{\varphi=1}.
\end{tikzcd}$$

Lastly we need to show the top horizontal arrow is an isomorphism. We may localize in the arc-topology, reducing to the case of Spf of a perfectoid ring S, which follows from applying Artin–Schreier–Witt and the fact that perfection does not change the étale site (of a characteristic p scheme). □

In order to pass from the derived statement above to concrete cohomology groups, we need the following:

Lemma 4.11. *For any i and n, the $\mathbb{Z}_p$-linear operator $\varphi-1$ is surjective on both* $\mathrm{H}^i\big((\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^{\wedge}/p^n\big)$ *and* $\mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)$.

Proof. We observe that, since $\varphi([a]) = [a]^p$ for any $a \in \mathfrak{m}_C^\flat$, we know φ acts topologically nilpotently on

$$[\mathfrak{m}_C^\flat] \cdot \mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big).$$

Therefore to check surjectivity of $\varphi-1$ on $\mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)$ we may quotient out $W(\mathfrak{m}_C^\flat)\cdot\mathrm{H}^i\big(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)$. Since $\mathcal{X}$ is smooth and proper over $\mathcal{O}_K$, we know the relevant groups are finitely generated modules over $W(C^\flat)$ and $W(\overline{k})$. Both of $C^\flat$ and $\overline{k}$ are algebraically closed field of characteristic p, hence we are reduced to [CL98, Exposé III, Lemma 3.3]. □

Using the same proof, we may identify p-adic étale cohomology of $\mathcal{X}_{\overline{k}}$ as Frobenius fixed points in various prismatic cohomology of $\mathcal{X}$, after suitably base changing to $W(\overline{k})$.

Porism 4.12. *Consider the $\mathfrak{S}$-algebra $W(\overline{k})[\![u]\!]$. We have an identification of G_k-modules:*

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \Big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n) \otimes_{\mathfrak{S}} W(\overline{k})[\![u]\!]\Big)^{\varphi=1} \cong \Big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n) \otimes_{\mathfrak{S}} W(\overline{k})[\![u]\!]\Big)^{\varphi=1}.$$

Proof. As showed in the proof of Lemma 4.11, we may compute Frobenius fixed points after quotient out $W(\mathfrak{m}^\flat_C)$ (for the A_{inf}-module) or u for the Frobenius module appearing in this porism. Now the first identification is reduced to Theorem 4.10 and an equality of $\mathfrak{S}$-algebras: $A_{\mathrm{inf}}/W(\mathfrak{m}^\flat_C) \cong W(\overline{k}) \cong W(\overline{k})[\![u]\!]/(u)$. The second identification is reduced to the fact that given a Frobenius module M on $W(\overline{k})$, then the natural map $M \to M \otimes_{W(\overline{k}),\varphi} W(\overline{k})$ given by $m \mapsto m \otimes 1$ induces an isomorphism of Frobenius fixed points. $\square$

Remark 4.13. Assume that the residue field $\overline{k}$ of $\mathcal{O}_K$ is separably closed. The above Porism 4.12 induces a map

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \Big(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n)\Big)^{\varphi=1} \hookrightarrow \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}_n) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

This map can be seen at the level of étale-sheaves on $\mathrm{fSch}_{/\,\mathrm{Spf}(\mathcal{O}_K)}$: $\mathbb{Z}_p/p^n \to \mathbb{\Delta}^{(1)}_n \to \mathcal{O}_{\mathcal{X}}/p^n$. Therefore we get a canonical map

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

In general, we just base change along $W(k) \to W(\overline{k})$ and get a G_k-equivariant map

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W(\overline{k}) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n) \otimes_W W(\overline{k}).$$

Later in Corollary 4.15 (3) we shall see a peculiar result concerning this map in the boundary degree. Now we come back to the relation between kernel of specialization map and u^∞-torsion in prismatic cohomology.

Theorem 4.14. *Let $\mathcal{X}$ be a smooth proper formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$. *Recall $\mathfrak{M}^i_n := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n)[u^\infty]$. There is a canonical isomorphism of G_k-modules*

$$\ker(\mathrm{Sp}^i_n) \cong \big(\mathfrak{M}^i_n \otimes_{\mathfrak{S}} A_{\mathrm{inf}}\big)^{\varphi=1} \cong \big(\mathfrak{M}^i_n/u \otimes_{W(k)} W(\overline{k})\big)^{\varphi=1}$$

for any $n \in \mathbb{N} \cup \{\infty\}$.

Proof. Combining Theorem 4.10 and Lemma 4.11, we get the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\mathrm{Sp}^i_n"] & \mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big) \arrow[r, "\varphi-1"] \arrow[d, "\mathrm{incl}"] & \mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big) \arrow[r] \arrow[d, "\mathrm{incl}"] & 0 \\
0 \arrow[r] & \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n) \arrow[r] & \mathrm{H}^i\big((\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge/p^n\big) \arrow[r, "\varphi-1"] & \mathrm{H}^i\big((\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge/p^n\big) \arrow[r] & 0.
\end{tikzcd}$$

We shall apply the snake lemma to the above. First, we claim

$$\mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty] \cong \ker(\mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big) \xrightarrow{\mathrm{incl}} \mathrm{H}^i\big((\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge/p^n\big).$$

When $n \in \mathbb{N}$ the map is localization with respect to ξ, hence tautological. We need to see this when $n = \infty$, namely we need to show injectivity of $\mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi]\big) \to \mathrm{H}^i(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})[1/\xi])^\wedge$. Here the latter completion is the classical p-adic completion: our assumption implies all cohomology groups $\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})$ have bounded p-torsion, hence derived p-completion agrees with derived p-completion. Since $\mathrm{H}^i_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})$ are finitely presented over A_{inf}, its localization with respect to ξ has separated p-adic topology, hence the p-adic completion map is injective.

Next, applying the base change property of prismatic cohomology to the p-completely faithfully flat map $\mathfrak{S} \to A_{\mathrm{inf}}$ and [BMS18, Proposition 4.3], we get an identification of Frobenius modules:

$$\mathfrak{M}^i_n \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \cong \mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty].$$

Now we get the first identification. To finish, just observe that $\varphi([a]) = [a]^p$, for any $a \in \mathfrak{m}^\flat_C$, which acts nilpotently on $\mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty]$. Hence the map $\varphi - 1$ is necessarily an isomorphism (of $\mathbb{Z}_p$-modules) on $[\mathfrak{m}^\flat] \cdot \mathrm{H}^i\big(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_C}/A_{\mathrm{inf}})/p^n\big)[\xi^\infty]$. Therefore we may quotient this part, as far as Frobenius fixed points are concerned, which leads to the second identification. $\square$

Corollary 4.15. *Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$ with ramification index e, let $i \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$. We have the following understanding of the kernel of the specialization map $\mathrm{Sp}^i_n \colon \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}/p^n)$.*

(1) If $e \cdot (i-1) < p-1$, then Sp^i_n is injective.
(2) If $e \cdot (i-1) < 2(p-1)$, then $\ker(\mathrm{Sp}^i_n)$ is annihilated by p^{i-1}.
(3) If $e \cdot (i-1) = p-1$, then $\ker(\mathrm{Sp}^i_n)$ is p-torsion, and corresponds to the étale-φ module $\mathfrak{M}^i_n$ over k. Moreover the natural G_k-equivariant map in Remark 4.13

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}_p} W(\overline{k}) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n) \otimes_W W(\overline{k})$$

induces a G_k-equivariant injection:

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \big(\mathcal{O}_K \otimes_W W(\overline{k})\big)/p \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n) \otimes_W W(\overline{k}).$$

Proof. All but the last statement immediately follow from Theorem 3.3 and Theorem 4.14. The last statement is a Galois-theoretic analog of Corollary 3.19. To prove this, we may base change $\mathcal{X}$ from $\mathcal{O}_K$ to $\mathcal{O}_K \otimes_W W(\overline{k})$ and it suffices to prove the statement there. Hence it suffices to assume that $\mathcal{O}_K$ has algebraically closed residue field $\overline{k}$.

Let us analyze the sequence of maps of $\mathfrak{S}$-modules

$$\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Z}/p^n) \cong \left(\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}^{(1)}_n)\right)^{\varphi=1} \hookrightarrow \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}^{(1)}_n) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

By Corollary 4.15 (3), we see the first map induces an isomorphism:

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \overline{k}[u]/(u^p) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, {\Delta}^{(1)}_n)[u^\infty].$$

The exact sequence $\mathrm{Fil}^1_{\mathrm{N}} \to {\Delta}^{(1)} \to \mathcal{O}_{\mathcal{X}}$ tells us that the kernel of the map

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \overline{k}[u]/(u^p) \to \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n)$$

is given by the induced first Nygaard filtration on the source, which we know is exactly u^e times the source, thanks to Proposition 3.17 (1). Notice that, as an $\mathcal{O}_K$-algebra, we have $\overline{k}[u]/(u^e) \cong \mathcal{O}_K/p$. Therefore we get the desired injection

$$\ker(\mathrm{Sp}^i_n) \otimes_{\mathbb{F}_p} \overline{k}[u]/(u^e) \hookrightarrow \mathrm{H}^i(\mathcal{O}_{\mathcal{X}}/p^n).$$

□

We refer readers to Section 6, especially Remark 6.10 and Remark 6.13 (4), for a related interesting example.

4.3. **Revisiting the integral Hodge–de Rham spectral sequence.**

In this subsection, we revisit the question discussed in [Li20]: what mild condition on $\mathcal{X}$ guarantees that the Hodge numbers of the generic fibre X can be read off from the special fibre $\mathcal{X}_0$?

Let us introduce a notation, which is the threshold of cohomological degree for which we can say something about the integral Hodge–de Rham spectral sequence, based on knowledge of the integral Hodge–Tate spectral sequence.

Notation 4.16. Let T be the largest integer such that $e \cdot (T-1) \leq p-1$.

The main result in this subsection is:

Theorem 4.17 (Improvement of [Li20, Theorem 1.1])**.** *Let $\mathcal{X}$ be a smooth proper p-adic formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$.*

(1) Assume there is a lift of $\mathcal{X}$ to $\mathfrak{S}/(E^2)$, then for all i, j satisfying $i + j < T$, we have equalities

$$h^{i,j}(X) = \mathfrak{h}^{i,j}(\mathcal{X}_0)$$

where the latter denotes virtual Hodge numbers of $\mathcal{X}_0$, defined as in [Li20, Definition 3.1].

(2) Assume furthermore that $e \cdot (\dim \mathcal{X}_0 - 1) \leq p-1$. Then the special fibre $\mathcal{X}_0$ knows the Hodge numbers of the rigid generic fibre X.

For instance, in the unramified case $e = 1$, condition (1) is automatic and condition (2) says we allow $\mathcal{X}$ to be at most dimension p. From the proof, we shall see that the Hodge numbers of X can be computed using the virtual Hodge numbers of $\mathcal{X}_0$ (see [Li20, Subsection 3.2]) together with Euler characteristics of $\Omega^i_{\mathcal{X}_0}$'s in an algorithmic way.

We largely follow the proof of [Li20, Theorem 1.1]. Just like there, we need to first analyze the integral Hodge–de Rham spectral sequence, hence the title of this subsection.

Theorem 4.18. *Let $\mathcal{X}$ be a smooth proper p-adic formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$ liftable to $\mathfrak{S}/(E^2)$. Let $n \in \mathbb{N} \cup \{\infty\}$.*

(1) The Hodge–de Rham spectral sequence for $\mathcal{X}_n$ has no nonzero differentials with source of total degree $< T$.

(2) If $e > 1$, then $\mathfrak{M}^T_n := \mathrm{H}^T_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty] = 0$. In particular, the prismatic cohomology $\mathrm{H}^m_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq M_m \otimes_{\mathbb{Z}_p} \mathfrak{S}$ is of the shape of a $\mathbb{Z}_p$-module M_m for all $m \le T$.

(3) If $e = 1$, The induced Hodge filtrations $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \subset \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X})$ are split for any $i \le p$ and any j.

(4) If $e > 1$, The induced Hodge filtrations $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \subset \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X})$ are split for any $i < T$ and any j.

Here $\mathcal{X}_n$ denotes the mod p^n fibre. We do not know if the split statement in (3) above holds at the mod p^n level. Mimicking the terminology in [Li20], we may say the Hodge–de Rham sequence for $\mathcal{X}_n$ is split degenerate up to degree T. We need some preparations.

Lemma 4.19.

(1) If $e = 1$, we have $\ell\big(\mathrm{Tor}^{\mathfrak{S}}_1(k, \mathcal{O}_K)\big) = \ell\big(\mathrm{Tor}^{\mathfrak{S}}_1(k, \varphi_{\mathfrak{S},*}\mathcal{O}_K)\big)$.

(2) If $e > 1$, we have $\ell\big(\mathrm{Tor}^{\mathfrak{S}}_1(k, \mathcal{O}_K)\big) < \ell\big(\mathrm{Tor}^{\mathfrak{S}}_1(k, \varphi_{\mathfrak{S},*}\mathcal{O}_K)\big)$.

(3) Let M be a finitely generated p^∞-torsion $\mathfrak{S}$-module without u-torsion, then

$$\ell\big(M \otimes_{\mathfrak{S}} \mathcal{O}_K\big) = \ell\big(M \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathcal{O}_K\big).$$

Here $\ell(-)$ denotes length of the $\mathcal{O}_K$-module.

Proof. For (1) and (2): Simply note that $\mathrm{Tor}^{\mathfrak{S}}_1(k, \mathcal{O}_K)$ is the u^e-torsion in $k = \mathfrak{S}/(p, u)$, whereas the module $\mathrm{Tor}^{\mathfrak{S}}_1(k, (\varphi_{\mathfrak{S}})_*\mathcal{O}_K)$ is the u^e-torsion in $k \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S} = \mathfrak{S}/(p, u^p)$.

For (3): It is easy to see that the condition guarantees a finite filtration on M with graded pieces given by $\mathfrak{S}/p \cong k[\![u]\!]$. Indeed we just make an induction on the exponent of powers of p that annihilates M and contemplate with the sequence

$$0 \to M[p] \to M \to M/M[p] \to 0.$$

Hence the equality of lengths follows from the equality of $\mathfrak{S}/p \otimes_{\mathfrak{S},\varphi_{\mathfrak{S}}} \mathfrak{S} \simeq \mathfrak{S}/p$. $\square$

Lemma 4.20. *Let $F \subset M$ be an inclusion of finitely generated $W(k)$-modules. If the induced maps $F/p^n \to M/p^n$ are injective for any $n \in \mathbb{N}$, then F is a direct summand in M.*

Proof. Denote M/F by C, the condition implies that $M[p^n] \twoheadrightarrow C[p^n]$ for all n. Write the torsion submodule C_{tor} as direct sums of cyclic torsion $W(k)$-modules, and use the condition, we see that each cyclic summand admits a section back to M. This way we see that the extension class restricted to 0 in $\mathrm{Ext}^1_{W(k)}(C_{\mathrm{tor}}, F)$, hence it must come from a class in $\mathrm{Ext}^1_{W(k)}(C/C_{\mathrm{tor}}, F)$. But now C/C_{tor} is finitely generated torsion free $W(k)$-module, which is well-known to be free $W(k)$-module, hence the extension group is 0. $\square$

Proof of Theorem 4.18. Let us show (1) and (2). The case of $n = \infty$ follows from the finite n case: for (1) this is by left exactness of taking inverse limit, for (2) this follows from Proposition 2.6. Now we assume $n \in \mathbb{N}$, the degeneration statement is equivalent to equality of lengths

$$\ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)) = \sum_{i+j=m} \ell(\mathrm{H}^{i,j}(\mathcal{X}_n)),$$

for any $m < T$. Note that by the mere existence of the Hodge–de Rham spectral sequence, we have the inequality

$$\ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)) \le \sum_{i+j=m} \ell(\mathrm{H}^{i,j}(\mathcal{X}_n))$$

for free for any m. Below we shall try to show the converse inequality for $m < T$.

To that end, by the same argument as in the first paragraph of [Li20, Proof of Theorem 1.1], the liftability condition implies that the Hodge–Tate spectral sequence degenerates up to degree $p-1$, see [BS19, Remark 4.13 and Proposition 4.14], [ALB19, Proposition 3.2.1], and [LL20, Corollary 4.23]. In particular, since $T-1 \leq p-1$ we have a splitting of $\mathcal{O}_K$-modules: $\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}_n) \simeq \bigoplus_{i+j=m} \mathrm{H}^{i,j}(\mathcal{X}_n)$ for any $m < T$. Here the Hodge–Tate cohomology of $\mathcal{X}_n$ is defined to be the quasi-syntomic cohomology of the mod p^n of the Hodge–Tate sheaf $\overline{\mathcal{O}}_{\mathbb{\Delta}}$. What remains to be shown is an inequality of length:

$$\ell(\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}_n)) \leq \ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)).$$

By the Hodge–Tate and de Rham comparisons of prismatic cohomology [BS19, Theorem 4.10 and Corollary 15.4], we have equalities:

$$\ell(\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}_n)) = \ell\big(\mathrm{H}^m_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n) \otimes_{\mathfrak{S}} \mathcal{O}_K\big) + \ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{m+1}, \mathcal{O}_K)\big)$$

and

$$\ell(\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X}_n)) = \ell\big(\mathrm{H}^m_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}_n) \otimes_{\mathfrak{S}, \varphi_{\mathfrak{S}}} \mathcal{O}_K\big) + \ell\big(\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{m+1}, (\varphi_{\mathfrak{S}})_* \mathcal{O}_K)\big).$$

Now the desired inequality between length of Hodge–Tate and de Rham cohomology follows from the definition of T, the inequality $m < T$, the Theorem 3.3, and the Lemma 4.19. This finishes the proof of (1).

Next we turn to (2), note that by Theorem 3.3 (3), if $\mathfrak{M}_n^T$ were nonzero, it would necessarily be a direct sum of k as an $\mathfrak{S}$-module. Then Lemma 4.19 (2) shows that when $e > 1$, the strict inequality

$$\ell(\mathrm{H}^{T-1}_{\mathrm{HT}}(\mathcal{X}_n)) < \ell(\mathrm{H}^{T-1}_{\mathrm{dR}}(\mathcal{X}_n))$$

holds, which violates the fact that the left hand side is the same as sum of length of Hodge cohomology groups whereas the right hand side is at most that sum. Hence we arrive at a contradiction. The vanishing of $\mathfrak{M}_n^m$ when $m < T$ already follows from Theorem 3.3 (1). The statement concerning structure of prismatic cohomology now follows from Proposition 2.6.

Now we turn to (3): $e = 1$, hence $T = p$. In this case, the statement (1) we proved above implies that for any $i \leq p$ and any j, the map $\mathrm{H}^i(\mathcal{X}_n, \mathrm{Fil}^j_{\mathrm{H}}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_n)$ is injective. Hence the submodule $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \subset \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X})$ has the property that it induces an injection modulo any p^n. The desired splitness follows from Lemma 4.20.

Lastly we show (4): when $e > 1$. We follow the argument of [Li20, Corollary 3.9]. Using the vanishing statement established in (2), it follows that we have abstract isomorphism $\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X}) \simeq \mathrm{H}^m_{\mathrm{dR}}(\mathcal{X})$ whenever $m < T$. Hence the argument of loc. cit. shows that in the range $m < T$, splitting of the Hodge–Tate filtration on $\mathrm{H}^m_{\mathrm{HT}}(\mathcal{X})$ is equivalent to the splitting of the Hodge filtration on $\mathrm{H}^m_{\mathrm{dR}}(\mathcal{X})$. We can then finish our proof, as liftability to $\mathfrak{S}/(E^2)$ gives the desired splitting of the Hodge–Tate filtration in the range $m < T \leq p$. ☐

Remark 4.21. Comparing our Theorem 4.18(1) with what Fontaine–Messing obtained [FM87, II.2.7.(i)] (assuming the existence of a lifting over W), we seemingly get a stronger statement: namely loc. cit only claims degeneration statement when the differential has target of degree $< p$ whereas ours allow the differential has source of degree $< p$ (so the target can have degree p). However this is due to Fontaine and Messing not trying to squeeze their method to the most optimal, which is understandable given how many indices they needed to take care of. Indeed, their [FM87, II.2.6.(ii)] implies the map in next degree (following their notation) $\oplus_{r=1}^t \mathrm{H}^{m+1}(J_n^{[r]}) \to \oplus_{r=0}^t \mathrm{H}^{m+1}(J_n^{[r]})$ is injective, which can be used to strengthen their [FM87, II.2.7.(i)], hence also gaining the extra degeneration statement we obtained here.

Now it is time to prove the main theorem in this subsection.

Proof of Theorem 4.17. Fix an $m < T$, and a $j \in \mathbb{N}$. We consider the map of two $\mathcal{O}_K$-complexes

$$\mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K).$$

Our Theorem 4.18 (1), (3) and (4) implies that this map in degree m satisfies the assumption of [Li20, Lemma 2.16] (with our m being the n in loc. cit.). We finish the proof of (1) by combining the conclusion of [Li20, Lemma 2.16] with the definition of Hodge numbers of X and virtual Hodge numbers of $\mathcal{X}_0$.

The fact that (1) implies (2) is rather a brain teaser. In the Hodge diamond of X, all numbers below the middle row, which is the row with total degree given by $\dim(X)$ ($\leq T$ by assumption), are given by the corresponding virtual Hodge number of $\mathcal{X}_0$. Hodge symmetry implies that $\mathcal{X}_0$ also knows all numbers above the middle row. Now for the middle row, simply use the fact that Euler characteristic is locally constant for any flat family of coherent sheaves. ☐

For the rest of this subsection, let us specialize to the case of $e = 1$. Using knowledge on the Hodge–de Rham spectral sequence, we have a similar degeneration of the "Nygaard–Prism" spectral sequence up to cohomological degree p.

Theorem 4.22. *Assume $e = 1$ (so $\mathcal{O}_K = W$), and let $n \in \mathbb{N} \cup \{\infty\}$. The map*

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$$

is injective, when $i < p$ or $i = p, j \leq p - 1$.

Recall that when $i = p$ and $j \geq p$, kernels of these maps have been studied in Proposition 3.17 (3).

Proof. We shall induct on j, the case of $j = 0$ being trivial. We need to stare at the following diagram:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E) \arrow[r] \arrow[d, hook] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j+1}_{\mathrm{N}}/p^n) \arrow[r] \arrow[d] & \mathrm{H}^i(\mathcal{X}, \Omega^{\geq j+1}_{\mathcal{X}/W}/p^n) \arrow[d, hook] \\
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n) \otimes_{\mathfrak{S}} (E) \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n) \arrow[r] & \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_n/W_n).
\end{tikzcd}$$

The rows are exact as they are part of long exact sequences, coming from exact sequences of sheaves on $\mathcal{X}_{\mathrm{qSyn}}$. The right vertical arrow is injective for all $i \leq p$ thanks to Theorem 4.18 (1), note that $T = p$ as $e = 1$. The left vertical arrow is injective by induction hypothesis.

Let us first show the statement for $i < p$. Take an element in the kernel of the middle vertical arrow, by diagram chasing we see that the element comes from an element α in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E)$. Now it suffices to show the image of α in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n) \otimes_{\mathfrak{S}} (E)$ is zero. Lastly we note that the further image in $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$ is zero, therefore it suffices to know $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$ has no E-torsion, or equivalently it has no u-torsion, thanks to Theorem 3.3 (1).

Finally let us show the statement when $i = p$, and let $j + 1 \leq p - 1$. Argue as the previous paragraph, we are reduced to showing: Given an element $\beta_j \in \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)$ whose image γ in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n)$ is an E-torsion, then the image of $\beta_j \otimes E$ in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j+1}_{\mathrm{N}}/p^n)$ is already zero. To that end, we need the help of another diagram:

$$\begin{tikzcd}
\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E) \arrow[r] \arrow[d] & \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n) \arrow[d] \\
\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n) \otimes_{\mathfrak{S}} (E) \arrow[r] \arrow[d, hook] & \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{j+1}_{\mathrm{N}}/p^n) \arrow[d] \\
\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n) \otimes_{\mathfrak{S}} (E) \arrow[r] & \mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}_n).
\end{tikzcd}$$

Although it will not be used, we point out that two vertical arrows in the top square are both injective because of Proposition 3.17 (2). Since γ is an E-torsion, we know it is (u, p)-torsion, see Theorem 3.3 (3). Therefore we see γ is the image of a (u, p)-torsion β_{p-1} in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n)$ thanks to Proposition 3.17 (1). By induction hypothesis, we see the image of β_{p-1} in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{N}}/p^n)$ is precisely β_j. Now we are done as $E \cdot \beta_{p-1} = 0$ in $\mathrm{H}^p_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^{p-1}_{\mathrm{N}}/p^n)$. ☐

5. Crystalline cohomology in boundary degree

Notation 5.1. Throughout this section let us fix $n \in \mathbb{Z} \cup \{\infty\}$, and fix e, i such that $e \cdot i = p - 1$. Let S be the PD envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$, let $c_1 = \varphi(E)/p \in S^\times$. Denote $\mathfrak{S}_n := \mathfrak{S}/p^n$ and $S_n := S/p^n$. Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$. Let $\mathfrak{M} := \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n^{(1)})$, let $\mathcal{M} := \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{O}_{\mathrm{crys}}) \cong \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}_{-/\mathfrak{S}}/p^n)$, and finally let $V := \mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)[u^\infty]$. We use Frob_k to denote the Frobenius on k.

Recall that, by Theorem 3.3, the module $\mathfrak{M}$ is u-torsion free and the Frobenius $\mathfrak{S}$-module V is an étale φ-module over k. Also recall [BS19, Theorem 5.2] and [LL20, Theorem 3.5 and Lemma 7.16] that, we have a short exact sequence of Frobenius S-modules:

$$(\boxdot) \qquad 0 \to \mathfrak{M} \otimes_{\mathfrak{S}_n} S_n \to \mathcal{M} \to \mathrm{Tor}_1^{\mathfrak{S}_n}(V, \varphi_* S_n) \cong \mathrm{Tor}_1^{\mathfrak{S}_1}(V, \varphi_* S_1) =: \overline{M} \to 0,$$

where the last equality follows from the fact that $S_1 = \mathfrak{S}_1 \otimes^{\mathbb{L}}_{\mathfrak{S}_n} S_n$. Here by assumption on $\mathcal{X}$, we know $\mathfrak{M}$ is finitely generated over $\mathfrak{S}$ and we can replace completed tensor with tensor to ease notation a little bit.

Let us give a functorial description of $\overline{M}$.

Lemma 5.2. *Let N be an $\mathfrak{S}_1$-module, then we have identifications of $\mathfrak{S}_1$-modules:*

(1) $\mathrm{Tor}_1^{\mathfrak{S}_1}(V, N) \cong V \otimes_k \big(N[u]\big)$*; and*

(2) $\mathrm{Tor}_1^{\mathfrak{S}_1}(V, \varphi_* N) \cong \mathrm{Frob}_k^*(V) \otimes_k \big(N[u^p]\big)$.

Here the $\mathfrak{S}$-module structures on right hand sides are via the second factor.

In particular we have $\overline{M} \cong \mathrm{Frob}_k^*(V) \otimes_k S_1[u^p]$.

Proof. Let us prove (2) here as the proof of (1) follows a similar argument. Note that

$$V \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} N = V \otimes^{\mathbb{L}}_{k,id} k \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} N = V \otimes^{\mathbb{L}}_{k,id} k \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} \mathfrak{S}_1 \otimes^{\mathbb{L}}_{\mathfrak{S}_1} N.$$

Then one simply computes

$$k \otimes^{\mathbb{L}}_{\mathfrak{S}_1,\varphi} \mathfrak{S}_1 \cong \mathfrak{S}_1/u^p,$$

with k module structure via Frobenius on k. Therefore the above derived tensor becomes

$$\mathrm{Frob}_k^* V \otimes_k \mathrm{Tor}_1^{\mathfrak{S}_1}(\mathfrak{S}_1/u^p, N) \cong \mathrm{Frob}_k^* V \otimes_k \big(N[u^p]\big)$$

□

In the following we shall describe the induced filtrations, divided Frobenii and connections on all terms of the sequence $\boxdot$.

5.1. **Understand filtrations.** Recall [LL20, Theorem 4.1] (and references thereof) we have filtered isomorphisms:

$$\mathrm{R}\Gamma(\mathcal{X}, \mathrm{Fil}^\bullet_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \xrightarrow{\cong} \mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}/S, \mathcal{I}^\bullet_{\mathrm{crys}}).$$

By the above identification, we need to understand the Hodge filtration on the derived de Rham cohomology of $\mathcal{X}/\mathfrak{S}$.

Lemma 5.3. *We have the following.*

(1) The map $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta_n^{(1)}) \to \mathfrak{M}$ is injective.

(2) The map $\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \to \mathcal{M}$ is injective.

This fact has appeared in the proof of [LL20, Theorem 7.22], for the convenience of readers let us reproduce its proof below. The key point is that the inequality $e \cdot (i-1) < p-1$ implies the i-th prismatic cohomology is u-torsion free, which in turn guarantee injectivity.

Proof. By [LL20, Corollary 4.23] and diagram chasing, we know the kernel of

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta_n^{(1)}) \to \mathfrak{M}$$

surjects onto the kernel of

$$\mathrm{H}^i(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}/p^n) \to \mathcal{M}.$$

Hence it suffices to prove (1).

By [LL20, Lemma 7.8] we know the i-th divided Frobenius $\varphi_i \colon \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \Delta_n^{(1)}) \to \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n)$ is an isomorphism. Combining with Theorem 3.3 (1) we see that the cohomology of Nygaard filtration has no finite length sub-$\mathfrak{S}$-module. Finally [LL20, Proposition 7.12] says the kernel of the map in (1) must be a finite length sub-$\mathfrak{S}$-module, hence zero. □

Notation 5.4. We denote the image of above injections by $\mathrm{Fil}^i \mathfrak{M}$ and $\mathrm{Fil}^i \mathcal{M}$ respectively.

The submodule $\mathrm{Fil}^i \mathcal{M} \subset \mathcal{M}$ induces filtrations on the first and the third term in the sequence $\boxdot$. For instance

$$\mathrm{Fil}^i \left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) := \left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) \cap \mathrm{Fil}^i \mathcal{M}$$

where the intersection happens inside $\mathcal{M}$, and

$$\mathrm{Fil}^i \overline{M} := \mathrm{Im}(\mathrm{Fil}^i \mathcal{M} \to \overline{M}).$$

Let us investigate these filtrations.

Let $\mathcal{I}^{[i]} \subset S$ be the i-th PD filtration ideal, which is p-completely generated by $\geq i$-th divided powers of $E(u)$ in S. Note that the quotient $S/\mathcal{I}^{[i]}$ is p-torsion free, hence the ideal $\mathcal{I}_n^{[i]} := \mathcal{I}^{[i]}/p^n \subset S_n$ can be regarded as the i-th PD filtration ideal on S_n.

Recall [LL20, §4] that we have a commutative diagram of sheaves on $\left(\mathcal{O}_K\right)_{\mathrm{qSyn}}$:

$$\begin{tikzcd}
(E(u)^j) \otimes_{\mathfrak{S}} \mathbbl{\Delta}^{(1)} \arrow[r] \arrow[d] & \mathrm{Fil}^j_{\mathrm{N}} \mathbbl{\Delta}^{(1)} \arrow[r] \arrow[d] & \mathbbl{\Delta}^{(1)} \arrow[d] \\
\mathcal{I}^{[j]} \otimes_S \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r] & \mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r] & \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} .
\end{tikzcd}$$

Lemma 5.5. *The diagram above induces the following commutative diagram of sheaves on* $\left(\mathcal{O}_K\right)_{\mathrm{qSyn}}$*:*

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathrm{Fil}^j_{\mathrm{N}} \mathbbl{\Delta}^{(1)}}{(E(u)^j)\otimes_{\mathfrak{S}} \mathbbl{\Delta}^{(1)}} \arrow[r] \arrow[d] & \frac{\mathbbl{\Delta}^{(1)}}{(E(u)^j)\otimes_{\mathfrak{S}} \mathbbl{\Delta}^{(1)}} \arrow[r] \arrow[d] & \frac{\mathbbl{\Delta}^{(1)}}{\mathrm{Fil}^j_{\mathrm{N}} \mathbbl{\Delta}^{(1)}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}}{\mathcal{I}^{[j]} \hat{\otimes}_S \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}} \arrow[r] & \frac{\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}}{\mathcal{I}^{[j]} \hat{\otimes}_S \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}} \arrow[r] & \frac{\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}}{\mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}} \arrow[r] & 0
\end{tikzcd}$$

which has short exact rows, and vertical arrows are isomorphisms if $j \leq p$ and remains so after derived mod p^n.

Proof. The derived mod p^n statement follows from the fact that derived mod p^n is exact. It suffices to show two of the three vertical arrows are isomorphisms.

Using $\mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \cong S \hat{\otimes}_{\mathfrak{S}} \mathbbl{\Delta}^{(1)}$, see [BS19, Theorem 5.2] and [LL20, Theorem 3.5], the middle vertical arrow is identified with

$$\mathbbl{\Delta}^{(1)} \hat{\otimes}_{\mathfrak{S}} \left(\frac{\mathfrak{S}}{(E(u)^j)} \longrightarrow \frac{S}{\mathcal{I}^{[j]}}\right),$$

hence it suffices to note that the ring map $\frac{\mathfrak{S}}{(E(u)^j)} \to \frac{S}{\mathcal{I}^{[j]}}$ is an isomorphism.

The right vertical arrow is an isomorphism (thanks to [LL20, Corollary 4.23]). □

Proposition 5.6. *The map* $\mathrm{Fil}^i \mathcal{M} \to \overline{M}$ *is surjective. Hence* $\mathrm{Fil}^i \overline{M} = \overline{M}$*.*

Proof. We stare at the following map between long exact sequences:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{N}} \mathbbl{\Delta}^{(1)}/p^n) \arrow[r] \arrow[d] & \mathfrak{M} \arrow[r] \arrow[d, "\iota"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \left(\mathbbl{\Delta}^{(1)}/\mathrm{Fil}^i_{\mathrm{N}} \mathbbl{\Delta}^{(1)}\right)/p^n) \arrow[d, "\simeq"] \\
\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \arrow[r] & \mathcal{M} \arrow[r] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \left(\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/\mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}\right)/p^n)
\end{tikzcd}$$

Chasing diagram, we see that it suffices to show the top right horizontal arrow is a surjection. Indeed, granting the surjectivity assertion, we get that the summation map

$$\mathrm{Fil}^i \mathcal{M} \oplus \mathfrak{M} \to \mathcal{M}$$

is a surjection. Projection further to $\overline{M}$ kills the second factor above, hence we get the desired surjectivity.

Lastly we prolong the top long exact sequence:

$$\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, (\mathbb{\Delta}^{(1)}/\operatorname{Fil}^i_{\mathrm{N}} \mathbb{\Delta}^{(1)})/p^n) \to \mathrm{H}^{i+1}_{\mathrm{qSyn}}(X_n, \operatorname{Fil}^i_{\mathrm{N}} \mathbb{\Delta}^{(1)}/p^n) \xrightarrow{\iota} \mathrm{H}^{i+1}_{\mathrm{qSyn}}(\mathcal{X}, \mathbb{\Delta}^{(1)}/p^n).$$

We are reduced to showing ι is injective, which is exactly Proposition 3.17 (2): note that $e \cdot \big((i+1)-1\big) = e \cdot i = p-1$. $\square$

Using what is proved in the above proposition, we can also understand $\operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big)$. The diagram before Lemma 5.5 implies that we have a natural map $\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]} \to \operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big)$. Since the map $\mathbb{\Delta}^{(1)} \to \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$ of quasi-syntomic sheaves is filtered, we also have a natural map $\operatorname{Fil}^i \mathfrak{M} \to \operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big)$. We remind readers Notation 5.4 that the source denotes H^i of i-th mod p^n Nygaard filtration.

Proposition 5.7. *The summation map* $\operatorname{Fil}^i \mathfrak{M} \oplus \big(\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}\big) \to \operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big)$ *is surjective.*

Proof. Note that

$$\frac{\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n}{\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}} = \mathfrak{M} \otimes_{\mathfrak{S}_n} \frac{S_n}{\mathcal{I}_n^{[i]}} = \mathfrak{M}/(E^i).$$

In the last equality, we use the fact that $i < p$ implies $S_n/\mathcal{I}_n^{[i]} = \mathfrak{S}_n/(E^i)$. Therefore any element x in $\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n$ can be written as $x = y + z$ with $y \in \mathfrak{M}$ and z is in the image of $\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}$. Hence we have

$$\operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big) = \Big(\operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big) \cap \mathfrak{M}\Big) + \operatorname{Im}(\mathfrak{M} \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}).$$

It suffices to show

$$\operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big) \cap \mathfrak{M} = \operatorname{Fil}^i \mathfrak{M} \coloneqq \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \operatorname{Fil}^i_{\mathrm{N}} \mathbb{\Delta}^{(1)}/p^n),$$

which exactly follows from chasing the diagram in the proof of Proposition 5.6. $\square$

Corollary 5.8. *Let $e = 1$ and $i = p-1$. Then the triple $(\mathcal{M}, \operatorname{Fil}^i \mathcal{M}, \varphi_i)$ is an object in* $\mathrm{Mod}^{\varphi, p-1}_{S,\mathrm{tor}}$.

Proof. Note that the map $\operatorname{Fil}^i \mathcal{M} \to \mathcal{M}$ is injective by Lemma 5.3. We need to show admissibility, i.e. the image φ_i generates $\mathcal{M}$. To that end, we shall explain why both images of $\varphi_i\colon \operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big) \to \operatorname{Fil}^i\big(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\big)$ and $\overline{\varphi_i}\colon \operatorname{Fil}^i \overline{M} = \overline{M} \to \overline{M}$ generates the target. For the latter, it follows from the $e = 1$ case of Proposition 5.13. For the former, just note that the Nygaard filtration, a.k.a. $\operatorname{Fil}^i \mathfrak{M}$ (see Notation 5.4) already has its image of φ_i generating the module, thanks to [LL20, Lemma 7.8.(3)]. $\square$

5.2. **Compute divided Frobenius.** Next we discuss the divided Frobenius on Fil^i of terms in the sequence ⚁. We will use φ_i to denote the divided Frobenius on both Nygaard and Hodge filtrations, hopefully readers can tell them apart by looking at the source of the arrow to see which divided Frobenius we are using.

Recall [LL20, Remark 4.24] that when $j \leq p-1$, the semi-linear Frobenius φ on $\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$ becomes uniquely divisible by p^j when restricted to the sub-quasi-syntomic sheaf $\operatorname{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$ (c.f. [Bre98, p. 10]), which we denote by φ_j. The divided Frobenius on Nygaard and Hodge filtrations are related by:

$$\begin{tikzcd}
\operatorname{Fil}^j_{\mathrm{N}} \mathbb{\Delta}^{(1)} \arrow[r, "\varphi_j"] \arrow[d, "\iota"] & \mathbb{\Delta} \arrow[d, "1 \otimes c_1^j"] \\
\operatorname{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\varphi_j"] & \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \cong \mathbb{\Delta} \hat{\otimes}_{\mathfrak{S}, \varphi} S,
\end{tikzcd}$$

as one computes: $\frac{\varphi}{\varphi(E)^j} \cdot (\frac{\varphi(E)}{p})^j = \frac{\varphi}{p^j}$. Restricting further to $\mathcal{I}^{[j]} \hat{\otimes}_{\varphi, \mathfrak{S}} \mathbb{\Delta} \subset \operatorname{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}$, the divided Frobenius is related to the (semi-linear) prismatic Frobenius via:

$$\begin{tikzcd}
\mathcal{I}^{[j]} \hat{\otimes}_{\varphi, \mathfrak{S}} \mathbb{\Delta} \arrow[r, "\varphi_j \otimes \varphi"] \arrow[d, "\iota"] & S \hat{\otimes}_{\varphi, \mathfrak{S}} \mathbb{\Delta} \arrow[d, "\cong"] \\
\operatorname{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\varphi_j"] & \mathrm{dR}^{\wedge}_{-/\mathfrak{S}},
\end{tikzcd}$$

where the φ_j and φ on the top arrow are respectively the divided Frobenius on $\mathcal{I}^{[j]} \subset S$ and the semi-linear Frobenius on Δ. Since we assumed $e \cdot i = p-1$, in particular $i \leq p-1$. From the discussion, we immediately get the following.

Lemma 5.9. *Restricting the divided Frobenius* $\varphi_i \colon \mathrm{Fil}^i \mathcal{M} \to \mathcal{M}$ *to* $\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$*, the image lands in the submodule* $\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n$*.*

Proof. By the above discussion, we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Fil}^i \mathfrak{M} \oplus \left(\mathcal{I}_n^{[i]} \otimes_{\mathfrak{S}_n} \mathfrak{M}\right) \arrow[r] \arrow[d] & \mathfrak{M} \otimes_{\mathfrak{S}_n} S_n \arrow[d, "\iota"] \\ \mathrm{Fil}^i \mathcal{M} \arrow[r, "\varphi_i"] & \mathcal{M}, \end{tikzcd}$$

where the top arrow is given by $\left(\varphi_i \otimes c_1^i\right) \oplus \left(\varphi_i \otimes \varphi\right)$. Our claim follows from Proposition 5.7 which says the image of the left vertical arrow is precisely $\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$. $\square$

Consequently the divided Frobenius $\varphi_i \colon \mathrm{Fil}^i \mathcal{M} \to \mathcal{M}$ descends to a semi-linear map $\mathrm{Fil}^i \overline{M} = \overline{M} \to \overline{M}$ (see Proposition 5.6), which we refer to as the *residual divided Frobenius*. Our next task is to relate this residual divided Frobenius with the Frobenius on V.

To that end, we factorize the divided Frobenius on i-th Hodge filtration as:

$$(\boxdot) \qquad \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \xrightarrow{\alpha} \Delta \widehat{\otimes}_{\mathfrak{S}} \mathcal{I}^{[i]} \xrightarrow{\mathrm{id} \otimes \varphi_i} \Delta \widehat{\otimes}_{\mathfrak{S}} \varphi_* S.$$

Here α is S-linear and is defined at the level of sheaves in $\left(\mathcal{O}_K\right)_{\mathrm{qSyn}}$: Recall [LL20, Remark 4.24] that on the basis of large quasi-syntomic algebras, we know

$$\mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} = \sum_{0 \leq j \leq i} \mathcal{I}^{[i-j]} \hat{\otimes}_{\mathfrak{S}} \mathrm{Fil}^j_{\mathrm{N}} \Delta^{(1)}.$$

Therefore the linear Frobenius

$$\Delta \widehat{\otimes}_{\mathfrak{S}} \varphi_* S \cong \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \xrightarrow{\beta} \Delta \widehat{\otimes}_{\mathfrak{S}} S$$

restricted to the i-th Hodge filtration lands in $\Delta \widehat{\otimes}_{\mathfrak{S}} \mathcal{I}^{[i]}$, and compose further with the i-th divided Frobenius on the second factor gives the semi-linear divided Frobenius.

Lemma 5.10. *The map* $\mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \xrightarrow{\alpha} \Delta \widehat{\otimes}_{\mathfrak{S}} \mathcal{I}^{[i]}$ *induces a commutative diagram:*

$$\begin{tikzcd} \mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right) \arrow[r] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]} \arrow[d] \\ \mathrm{Fil}^i \mathcal{M} \arrow[r, "\alpha"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n \hat{\otimes}_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}). \end{tikzcd}$$

The content of this lemma is that when we first derived mod α by p^n, then take $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, -)$, and finally restrict it to the submodule $\mathrm{Fil}^i\left(\mathfrak{M} \otimes_{\mathfrak{S}_n} S_n\right)$, it lands in the submodule $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}, \Delta_n) \otimes_{\mathfrak{S}_n} \mathcal{I}_n^{[i]}$ of the target. This is proved exactly the same way as Lemma 5.9 so let us omit it. From the above lemma, we know the map α descends to a map

$$\mathrm{Fil}^i \overline{M} = \overline{M} = \mathrm{Frob}_k^*(V) \otimes_k S_1[u^p] \xrightarrow{\overline{\alpha}} \mathrm{Tor}_1^{\mathfrak{S}_n}(V, \mathcal{I}_n^{[i]}) = V \otimes_k \mathcal{I}_1^{[i]}[u].$$

Here we use $\mathcal{I}_n^{[i]} \otimes^{\mathbb{L}}_{\mathfrak{S}_n} \mathfrak{S}_1 = \mathcal{I}_1^{[i]}$ and Lemma 5.2 (1) to obtain the identification of target.

Proposition 5.11. *Let* $F \colon V \to V$ *denote the semi-linear prismatic Frobenius on* V*, which induces linearized Frobenius* $\widetilde{F} \colon \mathrm{Frob}_k^*(V) \to V$*. Then the map*

$$\mathrm{Frob}_k^*(V) \otimes_k S_1[u^p] \xrightarrow{\overline{\alpha}} V \otimes_k \mathcal{I}_1^{[i]}[u]$$

is given by $\widetilde{F} \otimes u^{p-1}$*.*

Note that given a u^p-torsion in S_1, multiplying with u^{p-1} gives us a u-torsion in S_1, implicitly in the statement we have used the fact that the inclusion $\mathcal{I}_1^{[i]}[u] \subset S_1[u]$ is a bijection because $i \leq ep-1$.

Proof. We stare at the following commutative diagram of sheaves on $\left(\mathcal{O}_K\right)_{\mathrm{qSyn}}$:

$$\begin{tikzcd} \mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\alpha"] \arrow[d] & \Delta \widehat{\otimes}_{\mathfrak{S}} \mathcal{I}^{[i]} \arrow[d] \\ \Delta \widehat{\otimes}_{\mathfrak{S}} \varphi_* S \cong \mathrm{dR}^{\wedge}_{-/\mathfrak{S}} \arrow[r, "\beta"] & \Delta \widehat{\otimes}_{\mathfrak{S}} S \end{tikzcd}$$

which induces the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fil}^i \overline{M} \arrow[r, "\overline{\alpha}"] \arrow[d, "\cong"'] & V \otimes_k \mathcal{I}_1^{[i]}[u] \arrow[d, "\cong"] \\ \overline{M} \arrow[r, "\beta"] & V \otimes_k S_1[u]. \end{tikzcd}$$

The left vertical arrow is an isomorphism. As explained right after the statement, the right vertical arrow is also an isomorphism. Therefore we are reduced to computing the effect on H^{-1} of the map

$$(V \otimes^{\mathbb{L}}_{\mathfrak{S}_1, \varphi} \mathfrak{S}_1) \otimes^{\mathbb{L}}_{\mathfrak{S}_1} S_1 \to V \otimes^{\mathbb{L}}_{\mathfrak{S}_1} S_1$$

induced by the linearized Frobenius $V \otimes^{\mathbb{L}}_{\mathfrak{S}_1, \varphi} \mathfrak{S}_1 \cong \mathrm{Frob}_k^*(V) \otimes_k \mathfrak{S}_1/u^p \xrightarrow{\widetilde{F} \otimes \mathrm{proj}} V \otimes_k \mathfrak{S}_1/u$. We can choose the following explicit resolution of the above map of $\mathfrak{S}_1$-modules:

$$\begin{tikzcd} \mathrm{Frob}_k^*(V) \otimes_k \mathfrak{S}_1 \arrow[r, "\mathrm{id} \otimes u^p"] \arrow[d, "\widetilde{F} \otimes u^{p-1}"'] & \mathrm{Frob}_k^*(V) \otimes_k \mathfrak{S}_1 \arrow[r, "\mathrm{id} \otimes \mathrm{proj}"] \arrow[d, "\widetilde{F} \otimes \mathrm{id}"'] & \mathrm{Frob}_k^*(V) \otimes_k \mathfrak{S}_1/u^p \arrow[d, "\widetilde{F} \otimes \mathrm{proj}"] \\ V \otimes_k \mathfrak{S}_1 \arrow[r, "\mathrm{id} \otimes u"] & V \otimes_k \mathfrak{S}_1 \arrow[r, "\mathrm{id} \otimes \mathrm{proj}"] & V \otimes_k \mathfrak{S}_1/u. \end{tikzcd}$$

Tensor the above with S_1 over $\mathfrak{S}_1$ and look at the induced map on H^{-1} yields the conclusion. $\square$

The effect of the second arrow in ⚁ is very easy to understand: we only need to understand the divided Frobenius $\varphi_i \colon \mathcal{I}_1^{[i]}[u] \xrightarrow{\varphi_i} S_1[u^p]$. Note that we assumed $e \cdot i = p-1$, hence $e = 1$ means $i = p-1$.

Lemma 5.12. *The S_1-module $\mathcal{I}_1^{[i]}[u] = S_1[u]$ is generated by u^{ep-1}, and we have*

$$\varphi_i(u^{ep-1}) = \begin{cases} c_1^{p-1} \in S_1 = S_1[u^p], & \text{when } e = 1 \\ 0, & \text{when } e > 1. \end{cases}$$

Proof. The description of $\mathcal{I}_1^{[i]}[u]$ is well-known. It follows from the explicit description of $\mathcal{I}_1^{[i]} \subset S_1$, given in the proof of Proposition 5.7.

Let us choose a lift of $u^{ep-1} \equiv E(u)^{p-1} \cdot u^{e-1}$ to $\mathcal{I}^{[i]}$ and compute

$$\varphi_i(E(u)^{p-1} \cdot u^{e-1}) = c_1^{p-1} \cdot p^{p-1-i} \cdot u^{ep-p}.$$

After reducing mod p, the right hand side is 0 if $0 < p-1-i$ which is equivalent to $e > 1$, and when $e = 1$, the right hand side is c_1^{p-1}. $\square$

Putting everything together, we arrive at the following:

Proposition 5.13. *The divided Frobenius $\mathrm{Fil}^i \mathcal{M} \to \mathcal{M}$ descends to a residual divided Frobenius*

$$\overline{\varphi_i} \colon \mathrm{Fil}^i \overline{M} = \overline{M} \to \overline{M}.$$

After identifying $\overline{M} \cong \mathrm{Frob}_k^*(V) \otimes_k S_1[u^p]$, we have

$$\overline{\varphi_i} = \begin{cases} F \otimes c_1^{p-1} \cdot \varphi_{S_1}, & \text{when } e = 1 \\ 0, & \text{when } e > 1. \end{cases}$$

Here we abuse notation a little bit by writing the induced Frobenius on $\mathrm{Frob}_k^*(V)$ still as F.

Proof. The first sentence is Lemma 5.9. As for the computation of the residual divided Frobenius, we look at the sequence Equation (⚁), which gives rise to

$$\mathrm{Frob}_k^*(V)\otimes_k S_1[u^p] \xrightarrow{\overline{\alpha}} V\otimes_k \mathcal{I}_1^{[i]}[u] \xrightarrow{\mathrm{id}\otimes\varphi_i} V\otimes_{k,\varphi} S_1[u^p].$$

Combining Proposition 5.11 and Lemma 5.12 yields the result. □

5.3. **The connection.** In [LL20, Subsection 5.1] we explained how one gets a natural connection on the derived de Rham complex relative to $\mathfrak{S}$. Consequently we see that there is a connection $\nabla\colon \mathcal{M}\to\mathcal{M}$ satisfying $\nabla(f\cdot m) = f'\cdot m + f\cdot\nabla(m)$ for any $f\in S$ and $m\in\mathcal{M}$. In this section, we shall see that in a strong sense there is a unique such connection. As a corollary, the connection ∇ preserves the sequence ⚁. Moreover the compatibility between ∇ and divided Frobenius [LL20, Subsection 5.2] will determine the residual connection on $\overline{M}$.

Notation 5.14. Let $S[\epsilon] \coloneqq S[x]/(x^2)$ and let $S\xrightarrow{\iota_1} S[\epsilon]$ and $S\xrightarrow{\iota_2} S[\epsilon]$ be two ring homomorphisms defined as $\iota_1(f) = f\otimes 1$ and $\iota_2(f) = f\otimes 1 + f'\otimes\epsilon$.

Proposition 5.15. *There is a unique $\mathbb{E}_\infty$-$S[\epsilon]$-algebra isomorphism $\mathrm{dR}^\wedge_{R/\mathfrak{S}}\otimes_{S,\iota_1}S[\epsilon]\to\mathrm{dR}^\wedge_{R/\mathfrak{S}}\otimes_{S,\iota_2}S[\epsilon]$ which reduces to identity modulo ϵ and is functorial in formally smooth $\mathcal{O}_K$-algebra R.*

Proof. One observes the formula $\nabla\mapsto\big(g(m\otimes 1) = m\otimes 1+\nabla(m)\otimes\epsilon\big)$ gives a bijection between functorial connections on $\mathrm{dR}^\wedge_{R/\mathfrak{S}}$ and said functorial isomorphisms. Therefore the existence follows from [LL20, Subsection 5.1].

To show uniqueness, we follow the same argument as in the proof of [LL20, Theorem 3.13]. First by left Kan extension and quasi-syntomic descent, it suffices to check the uniqueness when viewing both sides as quasi-syntomic sheaves of $S[\epsilon]$-algebras. Secondly, by the same argument in loc.c cit., one sees that restricting to the category of quasi-syntomic $\mathcal{O}_K$-algebras of the form $\mathcal{O}_K\langle X_j^{1/p^\infty}; j\in J\rangle$ for some set J determines such morphisms of $S[\epsilon]$-algebras. Finally, when $\widetilde{R} = \mathcal{O}_K\langle X_j^{1/p^\infty}; j\in J\rangle$, both of the source and the target are given by $S[\epsilon]\langle X_j^{1/p^\infty}; j\in J\rangle$, now we need to show $g(X)$ has to be X.

To that end, let us assume $g(X^{1/p^n}) = X^{1/p^n} + Y_n\otimes\epsilon$, then we compute $g(X) = g(X^{1/p^n})^{p^n} = (X^{1/p^n} + Y_n\otimes\epsilon)^{p^n}\equiv X$ modulo p^n. Therefore we conclude $g(X)-X$ is divided by arbitrary powers of p, hence must be 0 by p-adic separatedness of $S[\epsilon]\langle X_j^{1/p^\infty}; j\in J\rangle$. □

Remark 5.16. For any qcqs smooth formal scheme $\mathcal{Y}$ over $\mathrm{Spf}(\mathcal{O}_K)$, the crystal nature of $\mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{Y}/S)$ gives a connection on $\mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{Y}/S)$, see [BdJ11, p.2 and Lemma 2.8]. Note that although in loc. cit. the authors were talking about crystals in quasi-coherent modules, their argument works in our setting of crystals in perfect complexes as $\Omega^{1,pd}_{S/W}$ is finite free over S, so there is no subtlety when derived tensoring it. Consequently, one gets a connection on $\mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{Y}/S)$, and when identifying $\mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{Y}/S)\cong\mathrm{dR}^\wedge_{\mathcal{Y}/\mathfrak{S}}$, our Proposition 5.15 shows the "crystalline" connection agrees with our "derived de Rham" connection.

Below we explain yet another way to get the connection, via prismatic crystal nature of prismatic cohomology. Recall [BS21, Construction 7.13] that there is a cosimplicial prism $\big(\mathfrak{S}^{(\bullet)}, J^{(\bullet)}\big)\to\mathcal{O}_K\cong\mathfrak{S}^{(\bullet)}/J^{(\bullet)}$. Let $S^{(\bullet)}\to\mathcal{O}_K$ be the similarly defined cosimplicial ring obtained by taking divided power envelopes of $\mathfrak{S}^{\hat{\otimes}_W n}\twoheadrightarrow\mathcal{O}_K$ where $[n]\in\Delta$. Note that there is a map of these cosimplicial rings induced by the Frobenius $\varphi_{\mathfrak{S}}^{\otimes\bullet}\colon\mathfrak{S}^{\otimes\bullet}\to\mathfrak{S}^{\otimes\bullet}$, let us explicate this for $\bullet = 0,1$ as we will need it later:

$$\begin{tikzcd}
\mathfrak{S}\cong W[\![u]\!] \arrow[r, "\iota_1"] \arrow[d, "u\mapsto u^p"'] & W[\![u,v]\!]\{\frac{u-v}{E(u)}\}^\wedge\cong\mathfrak{S}^{(1)}\cong W[\![u,v]\!]\{\frac{u-v}{E(v)}\}^\wedge \arrow[d, "u\mapsto u^p"', "v\mapsto v^p"] & W[\![v]\!]\cong\mathfrak{S} \arrow[l, "\iota_2"'] \arrow[d, "v\mapsto v^p"] \\
S\cong W[\![u]\!]\langle\!\langle E(u)\rangle\!\rangle \arrow[r, "\iota_1"] & W[\![u,v]\!]\langle\!\langle E(u), u-v\rangle\!\rangle\cong S^{(1)}\cong W[\![u,v]\!]\langle\!\langle E(v), u-v\rangle\!\rangle & W[\![v]\!]\langle\!\langle E(v)\rangle\!\rangle\cong S \arrow[l, "\iota_2"']
\end{tikzcd}$$

where $\langle\!\langle - \rangle\!\rangle$ denotes p-completely adjoining divided powers of the designated elements. To see the middle arrow is well-defined we use the fact that $\varphi(E(u))$ and $\varphi(E(v))$ in $S^{(1)}$ is p times a unit, and adjoining $\varphi(u-v)/p$ as a δ-ring is the same as adjoining divided powers of $u-v$, see [BS19, Corollary 2.39].

Now for any p-adically smooth $\mathcal{O}_K$-algebra R, we have a functorial isomorphism of $\mathbb{E}_\infty$-$\mathfrak{S}^{(1)}$-algebras:

$$\Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\iota_1} \mathfrak{S}^{(1)} \cong \mathfrak{S}^{(1)} \hat{\otimes}_{\iota_2,\mathfrak{S}} \Delta_{R/\mathfrak{S}}$$

by base change of prismatic cohomology. Base change the above along the aforesaid map $\mathfrak{S}^{(1)} \to S^{(1)}$ (and use either [BS19, Theorem 5.2] or [LL20, Theorem 3.5]) identifies the left (resp. right) hand side with

$$\Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\iota_1} \mathfrak{S}^{(1)} \hat{\otimes}_{\mathfrak{S}^{(1)},\varphi} S^{(1)} \cong \Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\varphi} S \hat{\otimes}_{S,\iota_1} S^{(1)} \cong \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \hat{\otimes}_{S,\iota_1} S^{(1)}$$

respectively $S^{(1)} \hat{\otimes}_{\iota_2,S} \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}$. This gives rise another description of the "crystalline" connection:

Proposition 5.17. *The following diagram commutes functorially in the p-adically smooth $\mathcal{O}_K$-algebra R:*

$$\begin{tikzcd}
\Delta_{R/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S},\iota_1} \mathfrak{S}^{(1)} \arrow[r, "\cong"] \arrow[d, "\hat{\otimes}_{\mathfrak{S}^{(1)}} S^{(1)}"'] & \mathfrak{S}^{(1)} \hat{\otimes}_{\iota_2,\mathfrak{S}} \Delta_{R/\mathfrak{S}} \arrow[d, "\hat{\otimes}_{\mathfrak{S}^{(1)}} S^{(1)}"] \\
\mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \hat{\otimes}_{S,\iota_1} S^{(1)} \arrow[r, "\cong"] & S^{(1)} \hat{\otimes}_{\iota_2,S} \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} .
\end{tikzcd}$$

Proof. Base changing the top arrow along $\mathfrak{S}^{(1)} \to S^{(1)}$ gives a potentially different functorial isomorphism in the bottom. Therefore it suffices to show that there is no non-trivial automorphism of the quasi-syntomic sheaf of $S^{(1)}$-algebras $R \mapsto \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \hat{\otimes}_{S,\iota_1} S^{(1)}$. The same argument as in [LL20, Theorem 3.13] does the job. ☐

As a consequence, we know the sequence ⚀ is stable under the connection. In fact more generally we have the following.

Corollary 5.18. *For any $j \in \mathbb{N}$ and any $n \in \mathbb{N} \cup \{\infty\}$, the connection on $\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$ preserves the submodule $\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}/p^n) \otimes_{\mathfrak{S}_n} S_n$.*

Proof. Under the dictionary between connections and crystals [BdJ11, Lemma 2.8], we need to show the isomorphism (note that both of $\iota_i \colon S \to S^{(1)}$ are p-completely flat)

$$\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \otimes_{S_n,\iota_1} S_n^{(1)} \cong S_n^{(1)} \otimes_{\iota_2,S_n} \mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$$

preserves the submodule $\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta^{(1)}/p^n) \otimes_{\mathfrak{S}_n} S_n$. This immediately follows from the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta/p^n) \hat{\otimes}_{\mathfrak{S}_n,\iota_1} \mathfrak{S}_n^{(1)} \arrow[r, "\cong"] \arrow[d, "- \hat{\otimes}_{\mathfrak{S}_n^{(1)}} S_n^{(1)}"'] & \mathfrak{S}^{(1)} \hat{\otimes}_{\iota_2,\mathfrak{S}_n} \mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \Delta/p^n) \arrow[d, "S_n^{(1)} \hat{\otimes}_{\mathfrak{S}_n^{(1)}} -"] \\
\mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \hat{\otimes}_{S_n,\iota_1} S_n^{(1)} \arrow[r, "\cong"] & S_n^{(1)} \hat{\otimes}_{\iota_2,S_n} \mathrm{H}^j_{\mathrm{qSyn}}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n).
\end{tikzcd}$$

induced by Proposition 5.17. ☐

Therefore we see that there is a *residual connection* $\overline{\nabla} \colon \overline{M} \to \overline{M}$. Recall [LL20, Subsection 5.2] that the connection ∇ and divided Frobenius φ_i are related by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fil}^i \mathcal{M} \arrow[r, "\varphi_i"] \arrow[d, "E(u)\cdot\nabla"'] & \mathcal{M} \arrow[d, "c_1\cdot\nabla"] \\
\mathrm{Fil}^i \mathcal{M} \arrow[r, "u^{p-1}\varphi_i"] & \mathcal{M}.
\end{tikzcd}$$

Since all maps descend down to $\mathrm{Fil}^i \overline{M} = \overline{M}$, we have the following:

Proposition 5.19. *There is a commutative diagram:*

$$\begin{tikzcd} \overline{M} \arrow[r, "\varphi_i"] \arrow[d, "E(u)\cdot\overline{\nabla}"'] & \overline{M} \arrow[d, "c_1\cdot\overline{\nabla}"] \\ \overline{M} \arrow[r, "u^{p-1}\overline{\varphi_i}"] & \overline{M}. \end{tikzcd}$$

Consequently when $e = 1$, after identifying $\overline{M} \cong \mathrm{Frob}_k^*(V) \otimes_k S_1[u^p]$, we have $\overline{\nabla}(v \otimes 1) = v \otimes \mathrm{d}\log(c_1)$.

Here $\mathrm{d}\log(c_1) = \frac{c_1'}{c_1} = \frac{u^{p-1}}{c_1}$.

Proof. The existence of such a commutative diagram follows from the preceding discussion and the fact that both of φ_i and ∇ descends to $\overline{M}$ by Proposition 5.13 and Corollary 5.18 respectively.

Start with $v \otimes 1$ at the top left corner and compare the end results of the two routes, we arrive at an identity:

$$\overline{\nabla}(F(v) \otimes 1) \cdot c_1^p + F(v) \otimes (p-1)c_1^{p-1} \cdot c_1' = 0,$$

where we used the description of φ_i in Proposition 5.13. Now we use the fact that $\overline{M}$ is p-torsion and the fact that F is a bijection to yield the desired conclusion. ☐

Corollary 5.20. *Let $e = 1$ and $h = p-1$, then the quadruple $(\overline{M}, \mathrm{Fil}^{p-1}\overline{M}, \varphi_{p-1}, \nabla)$ is a Breuil module and there is a canonical isomorphism $T_S(\overline{M}) \xrightarrow{\cong} (V \otimes_{W(k)} W(\overline{k}))^{\varphi=1}$ of representation of G_K. In particular the resulting Galois representation $T_S(\overline{M})$ is the unramified $\mathbb{F}_p$-representation associated with the étale φ-module V.*

Proof. The first part of statement follows from Lemma 5.2, Proposition 5.13, and Proposition 5.19. To compute $T_S(\overline{M})$, let $I_+A_{\mathrm{crys}} \subset A_{\mathrm{crys}}$ be the ideal so that I_+A_{crys} contains $W(\mathfrak{m}_{\mathcal{O}_{\mathbf{C}}^\flat})$ and $A_{\mathrm{crys}}/I_+A_{\mathrm{crys}} = W(\overline{k})$. It is clear that $\varphi^n(a) \to 0$ for any $a \in I_+A_{\mathrm{crys}}$ and $I_+A_{\mathrm{crys}} \cap S = I_+$. By (2.17), $\overline{M} \otimes_S I_+A_{\mathrm{crys}}$ is stable under the G_K-action. So we have a canonical map of G_K-representations

$$T_S(\overline{M}) = (\mathrm{Fil}^h \overline{M} \otimes_S A_{\mathrm{crys}})^{\varphi_h=1} = (\overline{M} \otimes_S A_{\mathrm{crys}})^{\varphi_h=1} \to (\overline{M} \otimes_A A_{\mathrm{crys}}/I_+A_{\mathrm{crys}})^{\varphi_h=1} = (\overline{M}/I_+ \otimes_k \overline{k})^{\varphi_h=1}.$$

By Proposition 5.13, if we identity $\overline{M} = \mathrm{Frob}^*V \otimes_k S_1[u^p]$ then $\forall x \otimes 1 \in \overline{M}/I_+$, $\varphi(x \otimes 1) = F(x) \otimes a_0^{p-1}$ with $a_0 = E(0)/p \in W(k)^\times$. So $\varphi_h : \overline{M}/I_+ \to \overline{M}/I_+$ is bijective. Using that $\lim_{n\to\infty} \varphi^n(a) = 0, \forall a \in I_+A_{\mathrm{crys}}$, we conclude that the above map is an isomorphism $T_S(\overline{M}) \xrightarrow{\cong} (\overline{M}/I_+ \otimes_k \overline{k})^{\varphi_h=1}$ of G_K-representations. Finally, we have to check that $\overline{M}/I_+ \simeq \mathrm{Frob}^*V$ as φ-modules. Indeed $\mathrm{Frob}^*V \to \mathrm{Frob}^*V \otimes_k S_1[u^p]/I_+S = \overline{M}/I_+$ via $x \mapsto a_0(x \otimes 1)$ is the required isomorphism of φ-modules. ☐

5.4. **Fontaine–Laffaille and Breuil modules.** In this subsection we assume $e = 1$. For simplicity we pick the uniformizer p, but all results in this subsection hold true with any other uniformizer. We shall compare the two approaches of understanding étale cohomology, as a Galois representation, from linear algebraic data on certain crystalline cohomology, which are due to Fontaine–Messing–Kato, and Breuil–Caruso.

First we need a reminder on the filtered comparison between derived de Rham cohomology and crystalline cohomology, see [LL20, Theorem 4.1] and references thereof.

Remark 5.21. Let $\mathcal{X}$ be a smooth p-adic formal scheme over $\mathrm{Spf}(W)$. We have filtered isomorphisms:

$$\mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^\bullet_{\mathrm{H}} \mathrm{dR}^\wedge_{-/W}) \xrightarrow{\cong} \mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}/W, \mathcal{I}^\bullet_{\mathrm{crys}}),$$

and

$$\mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^\bullet_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \xrightarrow{\cong} \mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}/S, \mathcal{I}^\bullet_{\mathrm{crys}}).$$

In classical references by Fontaine–Messing, Kato and Breuil–Caruso, they were considering the right hand side objects of the above isomorphisms. However we will be thinking about the derived de Rham side, as it is compatible with various techniques developed by Bhatt–Morrow–Scholze and Bhatt–Scholze.

For the remaining of this subsection we let $\mathcal{X}$ be a quasi-compact quasi-separated p-adic formal scheme over $\mathrm{Spf}(W)$. At the derived level, we have the following comparisons:

Proposition 5.22. *Consider the diagram:*

$$\begin{tikzcd} \mathcal{X} \arrow[d] \arrow[dr] \arrow[drr] & & \\ \mathrm{Spf}(W) \arrow[r, "u\mapsto p"] & \mathrm{Spf}(\mathfrak{S}) \arrow[r] & \mathrm{Spf}(W). \end{tikzcd}$$

For any $n \in \mathbb{Z} \cup \{\infty\}$, we have

(1) *The canonical maps of p-complete cotangent complexes $\mathbb{L}^{\wedge}_{\mathcal{X}/W} \to \mathbb{L}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from right triangle) and $\mathbb{L}^{\wedge}_{W/\mathfrak{S}} \to \mathbb{L}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from left triangle) induces an isomorphism*

$$\mathbb{L}^{\wedge}_{\mathcal{X}/W}/p^n \oplus \big(\mathbb{L}^{\wedge}_{W/\mathfrak{S}} \hat{\otimes}_W \mathcal{O}_{\mathcal{X}}\big)/p^n \xrightarrow{\cong} \mathbb{L}^{\wedge}_{\mathcal{X}/\mathfrak{S}}/p^n,$$

functorial in $\mathcal{X}/W$.

(2) *The canonical filtered maps of p-complete de Rham complexes $\mathrm{dR}^{\wedge}_{\mathcal{X}/W} \to \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from right triangle) and $\mathrm{dR}^{\wedge}_{W/\mathfrak{S}} \to \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}$ (from left triangle) induces a filtered isomorphism*

$$\big(\mathrm{dR}^{\wedge}_{\mathcal{X}/W} \hat{\otimes}_W \mathrm{dR}^{\wedge}_{W/\mathfrak{S}}\big)/p^n \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}/p^n,$$

functorial in $\mathcal{X}/W$.

(3) *Moreover the identification in (2) is compatible with divided Frobenii φ_j on j-th filtration of both sides for any $j \leq p-1$.*

In case readers are worried that we do not put any smoothness assumption on $\mathcal{X}$, just notice that both sides of these equalities are left Kan extended from smooth $\mathcal{X}$'s, therefore it suffices to prove these statements for smooth affine $\mathcal{X}$'s. That said, we will prove the statement without the smoothness assumption as the proof just works in this generality.

Proof. The finitary n cases follow from the case of $n = \infty$. Henceforth, we assume $n = \infty$.

(1): This follows from exact triangle of cotangent complexes associated with a triangle of morphisms.

(2): Let $\mathcal{X}_{\mathfrak{S}} := \mathcal{X} \times_{\mathrm{Spf}(W)} \mathrm{Spf}(\mathfrak{S})$ be the base change. Then we have $\mathcal{X} \cong \mathcal{X}_{\mathfrak{S}} \times_{\mathrm{Spf}(\mathfrak{S})} \mathrm{Spf}(W)$. These objects fit in a commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}_{\mathfrak{S}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spf}(W) \arrow[r, "u\mapsto p"] & \mathrm{Spf}(\mathfrak{S}) \arrow[r] & \mathrm{Spf}(W). \end{tikzcd}$$

Using Künneth formula for derived de Rham complex we obtain a filtered isomorphism:

$$\mathrm{dR}^{\wedge}_{\mathcal{X}_{\mathfrak{S}}/\mathfrak{S}} \hat{\otimes}_{\mathfrak{S}} \mathrm{dR}^{\wedge}_{W/\mathfrak{S}} \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{\mathcal{X}/\mathfrak{S}}.$$

The base change formula for derived de Rham complex gives us a filtered isomorphism:

$$\mathrm{dR}^{\wedge}_{\mathcal{X}/W} \hat{\otimes}_W \mathfrak{S} \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{\mathcal{X}_{\mathfrak{S}}/\mathfrak{S}}.$$

In both filtered isomorphisms above we put derived Hodge filtration on the derived de Rham complex, and trivial filtration on the coefficient ring W and $\mathfrak{S}$. Combining these two filtered isomorphisms gives our desired filtered isomorphism.

(3): This just follows from the fact that the two maps in (2) is compatible with divided Frobenii. □

Remark 5.23. Since $\mathfrak{S} \xrightarrow{u\mapsto p} W$ is a complete intersection, the p-adic derived de Rham complex $\mathrm{dR}^{\wedge}_{W/\mathfrak{S}} \cong S$ is given by Breuil's ring S with the Hodge filtration given by divided powers of $(u-p)$ and the usual Frobenius $u \mapsto u^p$. Similarly the mod p^n derived de Rham complex is S/p^n with the induced filtration: This is because the rings S and $S/\mathcal{I}^{[j]}$ are all p-torsion free for any $j \in \mathbb{N}$.

To obtain consequences at the level of cohomology groups, we need the following abstract lemma.

Lemma 5.24. *Let $\mathcal{C}$ be a stable ∞-category. Let $F\colon \mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}} \to \mathcal{C}$ be a map of simplicial sets. Then for any $0 < m \leq n$, we have a pushout diagram*

$$\begin{tikzcd} F(n+1-m,m) \arrow[r] \arrow[d] & F(n+1-m,m-1) \arrow[d] \\ \mathrm{colim}_{i+j\geq n, j\geq m} F(i,j) \arrow[r] & \mathrm{colim}_{i+j\geq n, j\geq m-1} F(i,j). \end{tikzcd}$$

In particular, the two inclusions $F(i,j) \to F(i-1,j)$ and $F(i,j) \to F(i,j-1)$ gives rise to a pushout diagram:

$$\bigoplus_{i+j=n+1, i>0, j>0} F(i,j) \rightrightarrows \bigoplus_{i+j=n} F(i,j) \longrightarrow \mathrm{colim}_{i+j\geq n} F(i,j).$$

Proof. The second statement follows from repeatedly applying the first statement and observing that

$$\mathrm{colim}_{i+j\geq n, j\geq n} F(i,j) = \mathrm{colim}_{(i,j)\geq(0,n)} F(i,j) = F(0,n)$$

as $(0,n)$ is the final object in $\{(i,j) \geq (0,n)\} \subset \mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$. The first statement follows from [Lur09, Proposition 4.4.2.2]: just apply the statement to

$$\{i+j \geq n, j \geq m-1\} = \{i+j \geq n, j \geq m\} \coprod_{\{(i,j)\geq(n+1-m,m)\}} \{(i,j) \geq (n+1-m, m-1)\}$$

yields the desired pushout diagram. □

Combining the previous two general statements yield the following.

Corollary 5.25. *Let $\mathcal{I}^{[\bullet]} \subset S$ be the filtration given by divided powers of $(u-p)$. The natural maps, for any $q+j \geq m$,*

$$\mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^q_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}) \hat{\otimes}_W \mathcal{I}^{[j]} \to \mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^m_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})$$

give rise to an exact triangle,

$$\bigoplus_{q+j=\ell+1, i>0, j>0} \mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^q_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})/p^n \hat{\otimes}_{W_n} \big(\mathcal{I}^{[j]}/p^n\big) \to$$

$$\to \bigoplus_{q+j=\ell} \mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^q_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})/p^n \hat{\otimes}_{W_n} \big(\mathcal{I}^{[j]}/p^n\big) \to \mathrm{R\Gamma}(\mathcal{X}, \mathrm{Fil}^n_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})/p^n$$

for any $\ell \in \mathbb{Z}$ and any $n \in \mathbb{Z} \cup \{\infty\}$.

Proof. The comparison of filtration Proposition 5.22 (2) shows the right hand side is given by the ℓ-th Day convolution filtration on $\mathrm{R\Gamma}(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})/p^n \hat{\otimes}_{W_n} \mathrm{dR}^{\wedge}_{W/\mathfrak{S}}/p^n$. Here the filtered ring $\mathrm{dR}^{\wedge}_{W/\mathfrak{S}}/p^n$ is given by $(S/p^n, \mathcal{I}^{[\bullet]})/p^n$, see Remark 5.23. Last we apply Lemma 5.24 to conclude the proof. □

Theorem 5.26 (c.f. [Bre98, p. 559 Remarques.(2)])**.** *For any $j, \ell \in \mathbb{Z}$ and any $n \in \mathbb{Z} \cup \{\infty\}$, use*

$$\mathrm{Im}\big(\mathrm{H}^j(\mathcal{X}, \mathrm{Fil}^{\ell}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \to \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}})/p^n\big) =: \mathrm{Fil}^{\ell}\, \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$$

to filter $\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$, and similarly filter $\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/W}/p^n)$. Then we have a filtered isomorphism

$$\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/W}/p^n) \hat{\otimes}_{W_n} \big(S/p^n\big) \xrightarrow{\cong} \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n).$$

Moreover it is compatible with the divided Frobenii φ_m on m-th filtration of both sides for all $m \leq p-1$.

Here again the ring S/p^n is equipped with the divided power ideal filtration. Concretely we have

$$\mathrm{Fil}^{\ell}\, \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) = \sum_{r+s=\ell} \mathrm{Fil}^r\, \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/W}/p^n) \hat{\otimes}_{W_n} \big(\mathcal{I}^{[s]}/p^n\big)$$

as sub-W_n-modules inside $\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \hat{\otimes}_{W_n} \big(S/p^n\big) \xrightarrow{\cong} \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$.

Proof. By Corollary 5.25, it suffices to show the exact triangle obtained induces short exact sequence after applying H^q. To that end, it suffices to show the map

$$\mathrm{H}^q(\mathcal{X}, \mathrm{Fil}^{\ell}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)\hat{\otimes}_{W_n}\big(\mathcal{I}^{[j+1]}/p^n\big) \to \mathrm{H}^q(\mathcal{X}, \mathrm{Fil}^{\ell}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)\hat{\otimes}_{W_n}\big(\mathcal{I}^{[j]}/p^n\big)$$

is injective for any q, j, ℓ, n. But this follows from the fact that $\big(\mathcal{I}^{[j]}/p^n\big)/\big(\mathcal{I}^{[j+1]}/p^n\big) \simeq W_n \cdot \gamma_j(u-p)$ is p-completely flat over W_n. The compatibility with divided Frobenii was checked in Proposition 5.22 (3). □

We arrive at the following result, which was already proved by Fontaine–Messing [FM87, Cor. 2.7] and Kato [Kat87, II.Proposition 2.5]. In fact they did not need the existence of a lift all the way to $\mathrm{Spf}(W)$.

Corollary 5.27. *Let $\mathcal{X}$ be a proper smooth p-adic formal scheme over W. Let $j \leq p-1$ and $n \in \mathbb{N}$. Then the natural map $\mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n)$ is injective, and the triple*

$$\Big(\mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n), \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i : \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n)\Big)$$

is an object in $\mathrm{FM}_{W(k)}$.

Proof. The injectivity follows from Theorem 4.18 (1). The triple tensored up to S is identified with

$$\Big(\mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n), \mathrm{H}^j(\mathcal{X}, \mathrm{Fil}^j_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n), \varphi_j\Big),$$

by Theorem 5.26. We have showed the map $\mathrm{H}^j(\mathcal{X}, \mathrm{Fil}^{\ell}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n) \to \mathrm{H}^j(\mathcal{X}, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/p^n)$ is injective, and the divided Frobenius φ_j generates the image: for $j \leq p-2$, this was the main result in our previous paper [LL20, Theorem 7.22 and Corollary 7.25]; and for $j = p-1$, use Lemma 5.3 and Corollary 5.8. Using the "if" part of Lemma 2.16, we see that $\Big(\mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n), \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[j]}_{\mathrm{crys}}), \varphi_i : \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^j_{\mathrm{crys}}(\mathcal{X}_n/W_n)\Big)$ is an object in $\mathrm{FM}_{W(k)}$. □

5.5. **Comparison to étale cohomology.** In this section, we study how crystalline cohomology $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n)$ compares to étale cohomology $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})$ in the boundary case $e \cdot i = p-1$. We shall freely use the notation and terminology from Section 2.

We first treat the case when $e = 1$ and $p-1$, in which case Corollary 5.27 shows that

$$M := \Big(\mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_n/W_n), \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_n/W_n, \mathcal{I}^{[p-1]}_{\mathrm{crys}}), \varphi_{p-1}\Big)$$

is an object in $\mathrm{FM}_{W(k)}$.

Theorem 5.28. *Notations as the above, then there exists a natural map $\eta\colon \mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \to T_{\mathrm{FM}}(M)$ of G_K-representations such that*

1. *The $\ker(\eta)$ is an unramified representation of G_K killed by p;*
2. *The $\mathrm{coker}(\eta)$ sits in a natural exact sequence $0 \to W \to \mathrm{coker}(\eta) \to W'$, where $W \cong \ker(\eta)$ and $W' \cong \ker(\mathrm{Sp}^{p-1}_n)$ is given by the kernel of specialization map in degree $(p-1)$.*

Note that by our Corollary 4.15 (3), $\ker(\mathrm{Sp}^{p-1}_n)$ is also an unramified G_K-representation killed by p. The $T_{\mathrm{FM}}(M)$ in the above theorem is what we meant by $\rho^{p-1}_{n,\mathrm{FL}}$ in Theorem 1.9.

Proof. Let $\mathfrak{M} := \mathrm{H}^{p-1}_{\Delta}(\mathcal{X}/\mathfrak{S}_n)$ (note that here we do not have Frobenius twist) and $\mathcal{M} = \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}/S_n)$. We have showed that the natural exact sequence ($\boxdot$) induces a natural exact sequence in $\mathrm{Mod}^{\varphi,p-1,\nabla}_{S,\mathrm{tor}}$:

$$0 \longrightarrow \underline{\mathcal{M}}(\mathfrak{M}) \longrightarrow \mathcal{M} \longrightarrow \overline{M} \longrightarrow 0\,,$$

see Proposition 5.6, Proposition 5.7, Proposition 5.13, Corollary 5.8, Corollary 5.18 and Proposition 5.19 for descriptions of the filtrations, Frobenii action, and connections. Furthermore, our Theorem 5.26 says $\mathcal{M} = \underline{\mathcal{M}}_{\mathrm{FM}}(M)$. Therefore, by left exactness of T_S, we have a natural sequence of G_K-representations:

$$0 \to T_S(\underline{\mathcal{M}}(\mathfrak{M})) \hookrightarrow T_S(\mathcal{M}) = T_{\mathrm{FM}}(M) \to T_S(\overline{M}).$$

On the other hand, we also have natural maps of G_K-representations:

$$\eta : \mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \xrightarrow{\sim} T_{\mathfrak{S}}(\mathfrak{M})(p-1) \xrightarrow[\alpha]{\sim} T^{p-1}_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\iota} T_S(\underline{\mathcal{M}}(\mathfrak{M})).$$

The first isomorphism is proved by [LL20, Cor. 7.4, Rem. 7.5]. As explained before Lemma 2.18, the map $\iota \circ \alpha$ is a map compatible with G_K-actions if the natural map $f : \mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to \mathcal{M}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ is compatible with G_K-actions on the both sides, where the G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ given by $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \simeq \mathrm{H}^{p-1}_{\Delta}(\mathcal{X}_{\mathcal{O}_{\mathbf{C}}}/A_{\mathrm{inf}})$ and the G_K-action on $\underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ is defined by formula (2.17). To prove that f is compatible with G_K-actions, note that the natural map $f' : \mathrm{H}^{p-1}_{\mathfrak{S}}(\mathcal{X}_{\mathcal{O}_{\mathbf{C}}}/A_{\mathrm{inf}}) \to \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_{\mathcal{O}_{\mathbf{C}}}/A_{\mathrm{crys}})$, which is compatible with G_K-actions, factors through $f : \mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to \underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ by using inclusion $\underline{\mathcal{M}}(\mathfrak{M}) \subset \mathcal{M}$ and isomorphism $\beta : \mathcal{M} \otimes_S A_{\mathrm{crys}} \simeq \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_{\mathcal{O}_{\mathbf{C}}}/A_{\mathrm{crys}})$. So it suffices to check that $\underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}} \to \mathcal{M} \otimes_S A_{\mathrm{crys}} \simeq \mathrm{H}^{p-1}_{\mathrm{crys}}(\mathcal{X}_{\mathcal{O}_{\mathbf{C}}}/A_{\mathrm{crys}})$ are compatible with G_K-actions. The compatibility of first map is due to that $\underline{\mathcal{M}}(\mathfrak{M}) \subset \mathcal{M}$ is stable under ∇ on $\mathcal{M}$ by Corollary 5.18, and the compatibility of second isomorphism is proved in [LL20, §5.3]. In summary, we obtain a natural map $\eta : \mathrm{H}^{p-1}_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(p-1) \to T_{\mathrm{FM}}(M)$ of G_K-representations.

Now we shall justify the two extra statements concerning kernel and cokernel of η. Since T_S is left exact, $\ker(\eta) \simeq \ker(\iota)$ which is unramified and killed by p, thanks to Corollary 2.20.

Easy diagram chase gives us a natural exact sequence:

$$0 \to \mathrm{coker}(\iota) \to \mathrm{coker}(\eta) \to T_S(\overline{M}).$$

By Corollary 2.20 we have $\mathrm{coker}(\iota) \cong \ker(\iota)$. The fact that $T_S(\overline{M}) \cong \ker(\mathrm{Sp}^{p-1}_n)$ follows from Corollary 5.20 and Theorem 4.14. ∎

Remark 5.29.

(1) From the proof, we see that the appearance of $\ker(\eta)$ and V is due to the defect of a key functor in integral p-adic Hodge theory, and the potential u-torsion in degree p (mod p^n) prismatic cohomology of $\mathcal{X}$ is to be blamed for the appearance of V'.

(2) It is unclear to us if the whole $\mathrm{coker}(\eta)$ is unramified and/or killed by p. It could even very well be the case that the sequence $0 \to W \to \mathrm{coker}(\eta) \to W'$ is split exact (in particular, right-exact) as G_K-representations. One would need extra input from integral p-adic Hodge theory, especially a further study of Breuil and Fontaine–Laffaille modules in the boundary degree case, in order to obtain such refinements.

Now we discuss the case $e > 1$ but $h \leq p-2$. We first recall that for $i \leq p-1$, in [LL20, §5.2] we have showed that $\mathcal{M}^i_n := (\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n), \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S_n, \mathcal{I}^{[i]}), \varphi_i)$ is an object in ${}^{\sim}\mathrm{Mod}^{\varphi,i}_S$. By the discussion before equation (7.24) in [LL20], we get the following exact sequence for $i \leq h \leq p-2$:

$$\cdots \mathrm{H}^{i-1}_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z}(h)) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[h]}_{\mathrm{crys}}) \xrightarrow{\varphi_h - 1} \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}), \tag{5.30}$$

let us mention that the crucial input is [AMMN21, Theorem F]. Thanks to $A_{\mathrm{crys},n}$ being flat over S_n, we have

$$\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[h]}_{\mathrm{crys}}) \cong \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[h]}_{\mathrm{crys}}) \otimes_S A_{\mathrm{crys}} \text{ and } \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \cong \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n) \otimes_S A_{\mathrm{crys}},$$

In this case, we can still define

$$T_S(\mathcal{M}^i_n) := \mathrm{Fil}^i(\mathcal{M}^i_n \otimes_S A_{\mathrm{crys}})^{\varphi_i = 1} = \ker\{\varphi_i - 1 : \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})\}.$$

The only difference is that the natural map $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is not expected to be injective without the condition $e \cdot i < p-1$.

Proposition 5.31. *Notation as above, we have a functorial isomorphism $T_S(\mathcal{M}^i_n) \cong \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z}(i))$.*

Proof. By (5.30), it suffices to show that $\varphi_i - 1 : \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) \longrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ is surjective for $i < p-2$. Choose an m large enough so that $\varphi_i(\mathrm{Fil}^m S_n) = 0$. So clearly $\varphi_i - 1$ restricted to the image of $\mathrm{Fil}^m S \otimes \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ is bijective. [9] Hence it suffices to show that

$$\varphi_i - 1 : \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) / \mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \longrightarrow \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) / \mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$$

[9] From now on, we abusively denote this image by $\mathrm{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$.

is surjective. Now we claim that both sides are finite generated $W_n(\mathcal{O}_{\mathbf{C}}^\flat)$-modules. Then the surjectivity of $\varphi_i - 1$ follows Lemma 5.34 below.

To check that both $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}})/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ and $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ are finitely generated over $W_n(\mathcal{O}_{\mathbf{C}}^\flat)$, it suffices to check that $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ and $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ are finite generated $\mathfrak{S}_n$-modules. This is clear for $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$: it is known that $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is a finite generated S_n-module, see [LL20, Proposition 7.19]. For $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$, consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^{i-1}_{\mathrm{qSyn}}(\mathcal{X}_n, {\mathbb{\Delta}}^{(1)}/\operatorname{Fil}^i_{\mathrm{N}} {\mathbb{\Delta}}^{(1)}) \arrow[r, "\alpha"] \arrow[d, "\wr"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \operatorname{Fil}^i_{\mathrm{N}} {\mathbb{\Delta}}^{(1)}) \arrow[r, "\beta"] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, {\mathbb{\Delta}}^{(1)}_{-/\mathfrak{S}}) \arrow[r] \arrow[d, "\iota"] & \cdots \\
\mathrm{H}^{i-1}_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{dR}^\wedge_{-/\mathfrak{S}}/\operatorname{Fil}^i_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \arrow[r, "\alpha'"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \operatorname{Fil}^i_{\mathrm{H}} \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \arrow[r, "\beta'"] & \mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{dR}^\wedge_{-/\mathfrak{S}}) \arrow[r] & \cdots
\end{tikzcd}$$

Since $\mathrm{H}^i_{\mathrm{qSyn}}(\mathcal{X}_n, \mathrm{dR}^\wedge_{\mathcal{X}_n/\mathfrak{S}_n})$ is finitely generated over S_n, the image of $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ inside $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is also finite $\mathfrak{S}_n$-generated. Here we have used the fact that $S_n/\operatorname{Fil}^m S_n$ is finitely generated over $\mathfrak{S}_n$. Note that $\ker(\beta') = \operatorname{Im}(\alpha')$ is also finitely generated over $\mathfrak{S}_n$. So $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})/\operatorname{Fil}^m S \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is finite $\mathfrak{S}_n$-generated. ☐

Lemma 5.32. *Let $C^\flat$ be a characteristic p algebraically closed complete non-Archimedean field, denote its ring of integers by $\mathcal{O}_C^\flat$ with maximal ideal $m^\flat$ and residue field $k^\flat$. Let M and N be two finitely generated $\mathcal{O}_C^\flat$ modules, let $F\colon M \to N$ be a Frobenius semilinear map, and let $G\colon M \to N$ be a linear map. The following are equivalent:*

1. *The map $F - G\colon M \to N$ is surjective;*
2. *The cokernel of $F - G\colon M \to N$ is finite;*
3. *The cokernel of $\overline{F - G}\colon M/m^\flat M \to N/m^\flat N$ is surjective;*
4. *The induced map $\overline{F - G}\colon M/m^\flat M \to N/m^\flat N$ is finite.*

Proof. It is clear that (1) $\implies$ (2), (3) $\implies$ (4). Below we shall show (4) $\implies$ (1).

Without loss of generality we may assume that both of M and N are finite free over $\mathcal{O}_C^\flat$. Indeed, let us choose maps from finite free modules, say P and Q, to M and N such that it is an isomorphism after modulo $m^\flat$. By Nakayama's lemma we see that these maps are surjective. Lift the two maps F and G, to get the following diagram

$$\begin{tikzcd}
P \arrow[r, "\widetilde{F}-\widetilde{G}"] \arrow[d] & Q \arrow[d] \\
M \arrow[r, "F-G"] & N.
\end{tikzcd}$$

By our choice of P and Q, condition (4) still holds for the top arrow. Since vertical arrows are surjective, it suffices to show that the top arrow is surjective. Therefore we may and do assume M and N are finite free.

Let us name the reduction of M and N by V and W which are finite dimensional $k^\flat$-vector spaces, and denote the reduction of F and G by f and g. We claim there are exhaustive increasing filtrations Fil_i with $0 \le i \le \ell$ on V and W respectively such that

- The maps f and g respect these two filtrations;
- The induced $f\colon \operatorname{Fil}_0 V \to \operatorname{Fil}_0 W$ is surjective;
- The induced $f\colon \operatorname{gr}_i V \to \operatorname{gr}_i W$ is 0 for all $1 \le i \le \ell$; and
- The induced $g\colon \operatorname{gr}_i V \to \operatorname{gr}_i W$ is an isomorphism for all $1 \le i \le \ell$.

To see the existence of such filtrations, we consider the following process: notice the image of $f\colon V \to W$ is a $k^\flat$ subspace, now look at the map $g\colon V \to W/\operatorname{Im}(f)$. By assumption of $\operatorname{Coker}(f-g)$ being finite, this map must be surjective, lastly we let

$$\operatorname{Fil}^0 V = \operatorname{Ker}(g\colon V \to W/\operatorname{Im}(f)),\ \operatorname{Fil}^0 W = \operatorname{Im}(f).$$

Replace V and W with $\mathrm{Fil}^0 V$ and $\mathrm{Fil}^0 W$ and repeat the above steps. This process terminates when we arrive at $\mathrm{Im}(f) = W$, and it will terminate as each time the dimension of W will drop. This way we get a decreasing filtration, after reversing indexing order we arrive at the desired increasing filtration.

Choose a sub-vector space $V_0 \subset \mathrm{Fil}_0(M/m^\flat M)$ on which f is an isomorphism, now lift the basis of $\mathrm{gr}_i(M/m^\flat M)$ for $1 \leq i \leq \ell$ and the basis of V_0 all the way to elements in M, we generate a finite free submodule $\widetilde{M}$. Now we contemplate the map $\widetilde{M} \to N$.

After choosing basis, we may regard both sides as $\mathcal{O}_C^\flat$ points of formal affine space over $\mathcal{O}_C^\flat$, and the map $F - G$ can be promoted to an algebraic map $h\colon \mathrm{Spf}(\mathcal{O}_C^\flat\langle \underline{X}\rangle) \to \mathrm{Spf}(\mathcal{O}_C^\flat\langle \underline{Y}\rangle)$. Note that by our choice of $\widetilde{M}$, these two formal affine spaces have the same dimension. Our choice of $\widetilde{M}$ guarantees that the reduction of h is finite, due to next lemma. Therefore the rigid generic fiber map h^{rig} is also finite by [BGR84, 6.3.5 Theorem 1], which implies it is flat by miracle flatness [Sta21, Tag 00R4], hence inducing a surjective map at the level of $C^\flat$-points.[10] □

The following lemma was used in the proof above, we thank Johan de Jong for providing an elegant proof.

Lemma 5.33. *Let k be a field, let $m > 1$ be an integer, and let $\left(a_{ij}\right)$ be an $n \times n$ matrix with entries in k. Let $\overline{h}\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be the morphism given by $\overline{h}^\sharp(y_i) = x_i^m + \sum_j a_{i,j} x_j$, then $\overline{h}$ is a finite morphism.*

Proof. This map can be compatified to a morphism between $\mathbb{P}^n_k$ preserving the infinity hyperplane. When restricted to the infinity hyperplane, the map becomes $[x_1 \colon \ldots \colon x_n] \mapsto [x_1^m \colon \ldots \colon x_n^m]$, which is non-constant. Lastly just observe that any endomorphism of $\mathbb{P}^n_k$ is either finite or constant. □

Here we have crucially used algebraically closedness of $\mathcal{O}_C^\flat$. Below is an example suggested to us by Johan de Jong illustrating the failure of (3) $\implies$ (4) when one drops the algebraically closed assumption. Start with the field $L_0 = \mathbb{F}_p(t^{1/p^\infty})$, pick a basis of $\mathrm{H}^1_{\text{ét}}(L_0, \mathbb{F}_p)$ we may find a (ginormous!) Galois pro-p infinite field extension L_1 such that the induced map on $\mathrm{H}^1_{\text{ét}}(-, \mathbb{F}_p)$ kills every basis vector except the first one. Repeat this process we arrive at a perfect field L such that $\mathrm{H}^1_{\text{ét}}(L, \mathbb{F}_p)$ is 1-dimensional over $\mathbb{F}_p$.

From the above we immediately conclude the following.

Lemma 5.34. *Let M and N be two finitely generated $A_{\inf}$ modules, let $F\colon M \to N$ be a Frobenius linear map and $G\colon M \to N$ be a linear map. Then the cokernel of $F - G$ (which is a $\mathbb{Z}_p$-linear map) is finitely generated over $\mathbb{Z}_p$ if and only if it is 0.*

Proof. The "if" part is trivial. For the "only if part": use right exactness of tensor and Lemma 5.32 we conclude that the cokernel is zero after modulo p. Now since finitely generated $\mathbb{Z}_p$ module is 0 if and only if its reduction modulo p is so, we get that the cokernel is zero. □

6. An example

Inspired by the example in [BMS18, Subsection 2.1], let us work out a direct generalization of their example (as suggested in [BMS18, Remark 1.3]) in this subsection. This example answers a question of Breuil [Bre02, Question 4.1] negatively.

Fix a positive integer n.[11] Let $\mathcal{E}_0$ be an ordinary elliptic curve over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{E}$ over $\mathrm{Spec}(W(k))$ be its canonical lift, in particular we have a closed immersion $\mu_{p^n} \subset \mathcal{E}[p^n]$ of finite flat group schemes over $\mathrm{Spec}(W(k))$. Let $\mathcal{O}_K \coloneqq W(k)[\zeta_{p^n}]$, choose $\zeta_{p^n} - 1$ to be the uniformizer in order to get $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$. To avoid confusion let us denote its Eisenstein polynomial as

$$E = d = \frac{(u+1)^{p^n} - 1}{(u+1)^{p^{n-1}} - 1} \in \mathfrak{S}.$$

On $\mathrm{Spec}(\mathcal{O}_K)$ we have the canonical group scheme homomorphism $\mathbb{Z}/p^n \to \mu_{p^n}$.

---

[10] Note that $C^\flat$-points of rigid generic fibre of an admissible formal scheme over $\mathcal{O}_C^\flat$ is the same as just $\mathcal{O}_C^\flat$-points of the formal scheme, see [Bos14, §8.3].

[11] We suggest first-time readers to simply take $n = 1$ which already has the "meat" and the notations and formulas become much simpler.

Construction 6.1. Let $\mathcal{X} \coloneqq [\mathcal{E}/(\mathbb{Z}/p^n)]$, a Deligne–Mumford stack which is smooth proper over $\mathrm{Spec}(\mathcal{O}_K)$. Here the action of $\mathbb{Z}/p^n$ on $\mathcal{E}$ is via μ_{p^n}. The generic fibre of $\mathcal{X}$ is the elliptic curve $\mathcal{E}_K/\mu_{p^n}$ (which in fact is isomorphic to $\mathcal{E}_K$ itself) and the special fibre is $\mathcal{E}_0 \times B(\mathbb{Z}/p^n)$. We have a factorization

$$\mathcal{E} \to \mathcal{X} \to \mathcal{E}/\mu_{p^n} \cong \mathcal{E}$$

of the lift of n-th Frobenius on $\mathcal{E}$ (exists because it is the canonical lift).

We want to understand the various cohomology theories of $\mathcal{X}$. Since all cohomology theories that we will encounter are étale sheaves and the quotient map $\mathcal{E} \to \mathcal{X}$, being a $\mathbb{Z}/p^n$-torsor, is a finite étale cover, we shall apply the Leray spectral sequence to this cover. Let us first record the structure of the prismatic cohomology of $\mathcal{E}_{\mathcal{O}_K}$ relative to $\mathfrak{S}$. We need the following lemma explicating the Frobenius operator on the (-1) Breuil–Kisin twist $\mathfrak{S}\{-1\}$, see [BL22, Section 2.2].

Lemma 6.2. *The Frobenius module $\mathfrak{S}\{-1\}$ has a generator x such that $\varphi(x) = E(u) \cdot \frac{p}{E(0)} \cdot x$.*

Proof. We know modulo u the Breuil–Kisin prism $\mathfrak{S}$ reduces to crystalline prism, whose (-1)-twist has a canonical generator $\overline{x}$ satisfying $\varphi(\overline{x}) = p \cdot \overline{x}$. Lifting this generator, we see that there is a generator x' of $\mathfrak{S}\{-1\}$ such that $\varphi(x') = a \cdot x'$ with $a \equiv p \bmod u$. On the other hand we know a is necessarily $E(u) \cdot \text{unit}$, due to [BL22, Construction 2.2.14]. Therefore we see that $a = E(u) \cdot \frac{p}{E(0)} \cdot v'$ where $v' \in \mathfrak{S}^\times$ and reduces to 1 mod u. It is a simple exercise that v' is of the form $\varphi(v)/v$ for some unit $v \in \mathfrak{S}^\times$ satisfying $v \equiv 1 \bmod u$ as well. Finally $x = x'/v$ is our desired generator. □

In our concrete situation, the Eisenstein polynomial d of $\zeta_{p^n} - 1$ has constant term p. Therefore our $\mathfrak{S}\{-1\}$ has a generator x such that $\varphi(x) = d \cdot x$.

Proposition 6.3. *We have isomorphism of Frobenius modules over $\mathfrak{S}$:*

(1) $\mathrm{H}^0_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}$;
(2) $\mathrm{H}^2_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}\{-1\}$; *and*
(3) $\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \simeq \mathfrak{S} \cdot \{e_1, e_2\}$ *with its Frobenius action given by* $\varphi(e_1) = e_1,$ *and* $\varphi(e_2) = a \cdot e_1 + d \cdot e_2$ *for some* $a \in \mathfrak{S}$.

Proof. It is well-known that elliptic curve has torsion-free crystalline cohomology. Therefore by Remark 3.6, we know all these prismatic cohomology groups are finite free $\mathfrak{S}$-modules.

The map $\mathcal{X} \to \mathrm{Spf}(\mathfrak{S}/d)$ always induces an isomorphism on H^0_{Δ} by Hodge–Tate comparison, this proves the first identification.

The second identification is well-known. For instance, the relative prismatic Chern class [BL22, Section 7.5] of (the line bundle associated with) the origin $0 \in \mathcal{E}_{\mathcal{O}_K}(\mathcal{O}_K)$ gives a map $c\colon \mathfrak{S}\{-1\} \to \mathrm{H}^2_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$. Reducing mod u this reduces to the first Chern class map in crystalline cohomology which is well-known to be an isomorphism. Since both source and target are finite free $\mathfrak{S}$-module, the map c is an isomorphism.

Cup product gives rise to a map of finite free Frobenius $\mathfrak{S}$-modules: $\bigwedge^2_{\mathfrak{S}} \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \to \mathrm{H}^2_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$. Modulo u this map reduces to the analogous map in crystalline cohomology which is again well-known to be an isomorphism, hence it is an isomorphism before mod u. Therefore it suffices to justify the existence of e_1. Since $\varphi(u) = u^p$, we see that

$$\left(\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})\right)^{\varphi=1} \cong \left(\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})/u\right)^{\varphi=1}.$$

Now we may use the crystalline comparison $\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})/u \cong \mathrm{H}^1_{\mathrm{crys}}(\mathcal{E}_0/W)^{(-1)}$ and the fact that $\mathcal{E}_0$ is ordinary to conclude the existence of e_1. □

Next let us compute the prismatic cohomology $\mathrm{H}^*_{\Delta}(\mathcal{X}/\mathfrak{S})$. We stare at the Leray spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathbb{Z}/p^n, \mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})) \Longrightarrow \mathrm{H}^{i+j}_{\Delta}(\mathcal{X}/\mathfrak{S})$$

which is compatible with Frobenius actions. In order to understand E_2 terms, we need the following:

Lemma 6.4. *The action of $\mathbb{Z}/p^n$ on $\mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$ is trivial.*

Proof. Let us use the p-completely flat base change $\mathfrak{S} \hookrightarrow W(C^\flat)$. Since our prismatic cohomology, as $\mathfrak{S}$-modules, are free, we get injections $\mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \hookrightarrow \mathrm{H}^j_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(C^\flat)$ compatible with the $\mathbb{Z}/p^n$-action. Using the étale comparison [BMS18, Theorem 1.8.(iv)], the target is canonically identified with $\mathrm{H}^j_{\text{ét}}(\mathcal{E}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$. We conclude the $\mathbb{Z}/p^n$-action on the target is trivial by comparing to the topological situation. $\square$

Therefore the second page, which is the starting page, of the above spectral sequence looks like

(⚃)
$$\begin{tikzcd}
\mathfrak{S}\{-1\} \arrow[drr] & 0 & \mathfrak{S}\{-1\}/p^n & \dots \\
\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \arrow[drr, "d_2"] & 0 & \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})/p^n & \dots \\
\mathfrak{S} & 0 & \mathfrak{S}/p^n & \dots
\end{tikzcd}$$

To our interest is the differential

$$d_2 \colon E_2^{0,1} = \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S}) \longrightarrow E_2^{2,0} \cong \mathfrak{S}/p^n.$$

Using the multiplicative structure of the spectral sequence this arrow determines the rest arrows, by degree reason the spectral sequence degenerates on the third page $E_3^{i,j} = E_\infty^{i,j}$.

Lemma 6.5. *The differential d_2 is divisible by u. In other words, it is zero after reduction modulo u.*

Proof. Let us look at the reduction modulo u of the spectral sequence ⚃, which is computing the crystalline cohomology of $\mathcal{X}/W$ by the crystalline comparison. Using the fact that $\mathrm{H}^2_{\mathrm{crys}}(B(\mathbb{Z}/p^n)/W) \cong W/p^n$ (see for instance [Mon21, Theorem 1.2]), we see that the d_2 modulo u must be zero. $\square$

Lemma 6.6. *We have $d_2(e_1) = 0$.*

Proof. This is because d_2 is Frobenius-equivariant. Now Proposition 6.3 (3) implies $e_1 \in \mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$ is fixed by Frobenius, yet Lemma 6.5 says its image under d_2 is divisible by u. So its image is divisible by arbitrary powers of u, hence must be zero. $\square$

Lemma 6.7. *After scaling e_2 by a unit in $\mathbb{Z}_p^\times$ we have $d_2(e_2) = (u+1)^{p^{n-1}} - 1$.*

Proof. Note that $\varphi(e_2) = d \cdot e_2$ by Proposition 6.3 (3), its image must be an element $x \in \mathfrak{S}/p^n$ satisfying the same Frobenius eigen-class condition. Next lemma guarantees that $d_2(e_2) = b \cdot ((u+1)^{p^{n-1}} - 1)$ for some $b \in \mathbb{Z}/p^n$. Étale comparison for prismatic cohomology says that base changing the spectral sequence ⚃ along $\mathfrak{S} \hookrightarrow W(C^\flat)$ gives a spectral sequence computing étale cohomology of $\mathcal{X}_C$ (base changed along $\mathbb{Z}_p \hookrightarrow W(C^\flat)$). Since $\mathcal{X}_C$ is an elliptic curve, its second étale cohomology has no torsion, hence the base changed d_2 must be surjective. In particular we see that $b \notin p \cdot \mathbb{Z}/p^n$, hence b must be a unit in $(\mathbb{Z}/p^n)^\times$. $\square$

In the proof above, we have utilized the following:

Lemma 6.8. *For any $m \leq n$, we have an exact sequence*

$$0 \to \mathbb{Z}/p^m \cdot \Big((u+1)^{p^{n-1}} - 1\Big) \to \mathfrak{S}/p^m \xrightarrow{\varphi - d} \mathfrak{S}/p^m.$$

Proof. First of all, let us check that $(u+1)^{p^{n-1}} - 1$ does satisfy the Frobenius action condition. Recall $d = \frac{(u+1)^{p^n}-1}{(u+1)^{p^{n-1}}-1}$, it suffices to know that $(u+1)^{p^n} \equiv (u^p+1)^{p^{n-1}}$ modulo p^n. When $n = 1$ this is well-known, induction on n proves the statement.

Next we verify this exact sequence for $m = 1$. In that situation $\mathfrak{S}/p \cong k[\![u]\!]$, and the Frobenius condition becomes $f^p = u^{p^{n-1}(p-1)} \cdot f$. One immediately verifies that $f \in \mathbb{F}_p \cdot u^{p^{n-1}}$.

Lastly we finish the proof by induction on m and applying the snake lemma to the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{S}/p^m \arrow[r, "\cdot p"] \arrow[d, "\varphi-d"] & \mathfrak{S}/p^{m+1} \arrow[r] \arrow[d, "\varphi-d"] & \mathfrak{S}/p \arrow[r] \arrow[d, "\varphi-d"] & 0 \\ 0 \arrow[r] & \mathfrak{S}/p^m \arrow[r, "\cdot p"] & \mathfrak{S}/p^{m+1} \arrow[r] & \mathfrak{S}/p \arrow[r] & 0. \end{tikzcd}$$

Notice that we have verified the kernel of middle vertical arrow surjects onto the kernel of right vertical arrow, thanks to the previous two paragraphs. The snake lemma tells us that the kernel of middle vertical arrow has length $m+1$, but we also know $\mathbb{Z}/p^{m+1} \cdot \Big((u+1)^{p^{n-1}} - 1\Big)$ sits inside the kernel. $\square$

From now on let us scale e_2 by the p-adic unit so that $d_2(e_2) = (u+1)^{p^{n-1}} - 1$. Using multiplicativity of the spectral sequence ⚃, we can compute the prismatic cohomology of $\mathcal{X}$. Let us record the result below.

Corollary 6.9. *The prismatic cohomology ring of $\mathcal{X}/\mathfrak{S}$ is*

$$\mathrm{H}^*_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathfrak{S}\langle e, f\rangle[g]/((u+1)^{p^{n-1}} - 1 \cdot g, p^n \cdot g, f \cdot g),$$

where e, f have degree 1 and are pulled back to e_1 and $p^n \cdot e_2$ respectively inside $\mathrm{H}^1_{\Delta}(\mathcal{E}_{\mathcal{O}_K}/\mathfrak{S})$, and g has degree 2 being the generator of $E_3^{2,0} = E_\infty^{2,0}$. Moreover the Frobenius action is given by

$$\varphi(e) = e,\ \varphi(f) = p^n a' \cdot e + d \cdot f,\ \text{and } \varphi(g) = g.$$

In particular we see that

$$\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty] \cong \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n) \cdot g$$

and

$$\mathrm{H}^\ell_{\Delta}(\mathcal{X}/\mathfrak{S}) \simeq \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n)$$

for all $\ell \geq 3$ generated by either $g^{\ell/2}$ or $e \cdot g^{(\ell-1)/2}$ depending on the parity of ℓ.

Here a' is a p-adic unit (that we used to scale e_2) times the constant a from Proposition 6.3 (3). We remark that g can be taken as a generator of the group cohomology $\mathrm{H}^2(\mathbb{Z}/p^n, \mathbb{Z}_p)$.

Later on we will produce a schematic example using approximations of $B(\mathbb{Z}/p^n)$, but before that let us observe that our stacky example matches with some predictions made in

Remark 6.10. The discussion in Section 4.1 extends to smooth proper Deligne–Mumford stacks, such as our $\mathcal{X}$. Since the generic fibre of $\mathcal{X}$ is $\left(\mathcal{E}/\mu_{p^n}\right)_K$, the map $g\colon \mathcal{X} \to$ Néron model of $\mathcal{X}_K$ becomes the natural map $[\mathcal{E}/(\mathbb{Z}/p^n)] \to \mathcal{E}/\mu_{p^n}$. Taking special fibre and factoring through $\mathrm{Alb}(\mathcal{X}_0) = \mathcal{E}_0$, we see the map f becomes the natural quotient map $\mathcal{E}_0 \to \mathcal{E}_0/\mu_{p^n}$ which has kernel μ_{p^n}. Note that when $n = 1$, we have $e = p-1$, and our Corollary 4.6 (3) indeed predicts that $\ker(f)$ can be at worst a form of several copies of μ_p.

Since $\mathcal{X}_0 \cong \mathcal{E}_0 \times B(\mathbb{Z}/p^n)$, we know its π_1 is abelian with torsion given by $\mathbb{Z}/p^n$. Consequently the torsion part in $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_0, \mathbb{Z}_p)$ is also given by $\mathbb{Z}/p^n$. Since $\mathcal{X}_C$ is an elliptic curve, its étale cohomology is torsion-free. Hence the specialization map in degree 2 for p-adic étale cohomology has kernel given by $\mathbb{Z}/p^n$. This matches up with what Theorem 4.14 predicts. Indeed since $\varphi(g) = g$, we see that

$$\Big(\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]\Big)^{\varphi=1} = \Big(\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S})[u^\infty]/u\Big)^{\varphi=1} = (W/p^n \cdot g)^{\varphi=1} = \mathbb{Z}/p^n \cdot g.$$

Here in the second identification we have used the fact that u divides $(u+1)^{p^{n-1}} - 1$.

The above stacky example can be turned into a scheme example, by the procedure of approximation explained below.

Construction 6.11. Choose a representation V of $\mathbb{Z}/p^n$ over $\mathbb{Z}_p$, so that inside $\mathbb{P}(V)$ one can find a $\mathbb{Z}/p^n$-stable complete intersection 3-fold $\mathcal{Y}$ with no fixed point and smooth proper over $\mathbb{Z}_p$, see [BMS18, 2.7-2.9]. Now we form $\mathcal{Z} \coloneqq (\mathcal{E} \times_{\mathbb{Z}_p} \mathcal{Y})_{\mathcal{O}_K}/(\mathbb{Z}/p^n)$, which is a smooth proper relative 4-fold over $\mathcal{O}_K$. Here the action of $\mathbb{Z}/p^n$ is the diagonal action.

Let us show the prismatic cohomology of $\mathcal{Z}/\mathfrak{S}$ approximates that of $\mathcal{X}/\mathfrak{S}$ in degrees ≤ 2 in a suitable sense.

Proposition 6.12. *The natural $\mathbb{Z}/p^n$-equivariant projection $(\mathcal{E} \times_{\mathbb{Z}_p} \mathcal{Y})_{\mathcal{O}_K} \to \mathcal{E}_{\mathcal{O}_K}$ gives rise a map $\mathcal{Z} \to \mathcal{X}$, which induces isomorphisms:*

$$\mathrm{H}^0_{\Delta}(\mathcal{X}/\mathfrak{S}) \xrightarrow{\cong} \mathrm{H}^0_{\Delta}(\mathcal{Z}/\mathfrak{S}) \text{ and } \mathrm{H}^1_{\Delta}(\mathcal{X}/\mathfrak{S}) \xrightarrow{\cong} \mathrm{H}^1_{\Delta}(\mathcal{Z}/\mathfrak{S}).$$

Together with the similarly defined map $\mathcal{Z} \to \mathcal{Y}_{\mathcal{O}_K}/(\mathbb{Z}/p^n)$, we have

$$\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S}) \oplus \mathfrak{S}\{-1\} \xrightarrow{\cong} \mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S}).$$

Proof. We want to apply the Leray spectral sequence to the finite étale cover $(\mathcal{E} \times_{\mathbb{Z}_p} \mathcal{Y})_{\mathcal{O}_K} \to \mathcal{Z}$.

First we claim the natural embedding $\mathcal{Y}_{\mathcal{O}_K} \to \mathbb{P}(V)_{\mathcal{O}_K}$ induces an isomorphism of prismatic cohomology in degrees ≤ 2. It suffices to show the same for Hodge–Tate cohomology, which in turn reduces us to showing it for Hodge cohomology. This follows from $\mathcal{Y}$ being a smooth complete intersection inside $\mathbb{P}(V)$, see [ABM21, Proposition 5.3]. Lastly it is well-known that $\mathrm{H}^2_{\Delta}(\mathbb{P}(V)_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}\{-1\}$, for instance see [BL22, 10.1.6].

Since $\mathrm{H}^1_{\Delta}(\mathcal{Y}_{\mathcal{O}_K}/\mathfrak{S}) = 0$, the Leray spectral sequence in degrees ≤ 2 is the direct sum of the spectral sequences for $\mathcal{X}$ and $\mathcal{Y}/(\mathbb{Z}/p^n)$ respectively. This gives the statement for cohomological degrees ≤ 1. Looking at the shape of the Leray spectral sequence for $\mathcal{Y}/(\mathbb{Z}/p^n)$, one easily sees that the $E_2^{0,2}$ term:

$$\left(\mathrm{H}^2_{\Delta}(\mathcal{Y}_{\mathcal{O}_K}/\mathfrak{S})\right)^{\mathbb{Z}/p^n} \cong \mathrm{H}^2_{\Delta}(\mathcal{Y}_{\mathcal{O}_K}/\mathfrak{S}) \cong \mathfrak{S}\{-1\}$$

survives, hence proving the statement in cohomological degree 2. □

Remark 6.13.

(1) Since $\mathrm{H}^2_{\Delta}(\mathcal{X}/\mathfrak{S}) \oplus \mathfrak{S}\{-1\} \cong \mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S})$ we know the $\mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S})[u^\infty] \cong \mathfrak{S}/((u+1)^{p^{n-1}} - 1, p^n)$. In particular its annihilator ideal is $((u+1)^{p^{n-1}} - 1, p^n) \in \mathfrak{S}$, congruent to $(u^{p^{n-1}})$ modulo (p). The ramification index is $p^{n-1}(p-1)$, hence these examples demonstrate that the bound in Corollary 3.2 is sharp.

(2) Now assume $p \geq 3$, then $p - 2 + 1 \geq 2$, our previous result [LL20, Theorem 7.22] together with the fact that $\mathrm{H}^2_{\Delta}(\mathcal{Z}/\mathfrak{S})$ contains u-torsion implies that Breuil's first crystalline cohomology of $\mathcal{Z}$, with mod p^m coefficient for any m, together with Frobenius action and filtration is *not* a Breuil module. When $n = 1$, we have $e = p - 1$, which shows that our result [LL20, Corollary 7.25] is sharp. Below we shall see that the first crystalline cohomology cannot even support a strongly divisible lattice structure because it is torsion-free but not free.

(3) Same reasoning as in Remark 6.10 shows that the map $f\colon \mathrm{Alb}(\mathcal{Z}_0) \to \mathrm{Alb}(\mathcal{Z})_0$ is given by the quotient map $\mathcal{E}_0 \to \mathcal{E}_0/\mu_{p^n}$.

(4) The special fibre $\mathcal{Z}_0 = \mathcal{E}_0 \times (\mathcal{Y}_0/(\mathbb{Z}/p^n))$ has abelian π_1, with its torsion part being $\mathbb{Z}/p^n$. Here we used the fact that complete intersections of dimension ≥ 3 are simply connected, see [Sta21, Tag 0ELE]. On the other hand the same argument as in [BMS18, proof of Proposition 2.2.(i)] shows that $\pi_1(\mathcal{Z}_C) \cong \widehat{\mathbb{Z}}^{\oplus 2}$. Hence we see again the specialization map $\mathrm{H}^2_{\text{ét}}(\mathcal{Z}_0) \to \mathrm{H}^2_{\text{ét}}(\mathcal{Z}_C)$ has kernel given by $\mathbb{Z}/p^n$, c.f. [BMS18, Remarks 2.3-2.4].

In fact it was the desire of finding examples with non-trivial kernel under specialization, together with inspiring discussions with Bhatt and Petrov separately, that leads us to analyze and generalize the example in [BMS18, Subsection 2.1]. The Enriques surface used there turns out to be a little bit red herring, the actual purpose it serves is just an approximation of classifying stack of $\mathbb{Z}/2$, like our $(\mathcal{Y}/(\mathbb{Z}/p^n))$ here.

Finally let us explain why our example negates a prediction of Breuil [Bre02, Question 4.1]. Let S denote the p-adic PD envelope of $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$.

Proposition 6.14. *There is an exact sequence:*

$$0 \to \mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S) \hookrightarrow S \cdot \{e_1, e_2\} \xrightarrow{d_2} S/p^n,$$

where $d_2(e_1) = 0$ and $d_2(e_2) = (u+1)^{p^n} - 1$. In particular $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S)$ is torsion-free rank 2 but not *free unless $(n,p) = (1,2)$.*

Proof. In Proposition 6.12 we see that the map $\mathcal{Z} \to \mathcal{X}$ induces isomorphism in degree 1 prismatic cohomology and u^∞-torsion in degree 2 prismatic cohomology. The comparison between prismatic and crystalline cohomology [LL20, Theorem 1.5] (see also [BS19, Theorem 5.2]) tells us that $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{X}/S) \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S)$. The same comparison result implies that after applying $- \otimes_{\mathfrak{S}} \varphi_* S$ to the spectral sequence ⚃, one calculates the crystalline cohomology of $\mathcal{Z}/S$. Therefore the first statement follows from Lemma 6.6 and Lemma 6.7. Note that $\varphi\big((u+1)^{p^{n-1}} - 1\big) \equiv (u+1)^{p^n} - 1$ modulo p^n.

To see the second assertion, note that $\mathrm{H}^1_{\mathrm{crys}}(\mathcal{Z}/S) \cong S \cdot e_1 \oplus J \cdot e_2$ where J is the ideal

$$\{x \in S \mid p^n \text{ divides } x \cdot ((u+1)^{p^n} - 1)\}.$$

If J were free, then it would be generated by a particular such element, denoted below as g. Let $g = \sum_{i=0}^{\infty} a_i \frac{u^i}{e(i)!}$ with $a_i \in W(k)$ approaching 0 and $e(i) = \lfloor \frac{i}{e} \rfloor$ where $e = p^{n-1} \cdot (p-1)$, note that every element in S can be uniquely expressed in this form. Since p^n trivially lies in J, it must also be divisible by this g. Therefore there exists $h_1 \in S$ such that $gh_1 = p^n$. Write $q_n = (u+1)^{p^n} - 1$.

Claim 6.15. *a_0 is nonzero and divisible by p.*

Proof. The fact that a_0 is nonzero follows from $gh_1 = p^n$. If a_0 is a unit in $W(k)$ then $g \in S^\times$ is a unit, which implies $q_n \in p^n S$. But this is equivalent to $n = 1$ and $p = 2$. □

So now we can assume that $a_0 = pa_0'$ with $a_0' \neq 0$. Pick $\frac{u^m}{e(m)!}$ so that $\frac{u^m}{e(m)!} q_n \in p^n S$ (select $m = p^n e - 1$ for example). Then we have $gh_2 = \frac{u^m}{e(m)!}$ for some $h_2 \in S$. The above equation implies that $h_2 = \sum_{i=m}^{\infty} b_i \frac{u^i}{e(i)!}$. But compare u^m term on both sides, we have $a_0 b_m = 1$ which contradicts $a_0 = pa_0'$. This finishes the proof. □

Remark 6.16. In Breuil's terminology, this shows that the first crystalline cohomology of our examples are *not* strongly divisible lattices [Bre02, Definition 2.2.1]. This contradicts the claimed [Bre02, Theorem 4.2.(2)], in the proof of loc. cit. one is led to Faltings' paper [Fal99]. However Faltings was treating the case of p-divisible groups, hence Breuil's Theorem/proof should only be applied to abelian schemes. Now it is tempting to say smooth proper schemes over $\mathcal{O}_K$ and their Albanese should share the same H^1 for whatever cohomology theory.[12] But our example clearly negates this philosophy: the stacky example is squeezed between two abelian schemes and neither should really be the "mixed characteristic 1-motive" of our stack (even though these two abelian schemes are abstractly isomorphic). Indeed the sequence $\mathcal{E}_{\mathcal{O}_K} \to \mathcal{X} \to \mathcal{E}_{\mathcal{O}_K}$ has the property that the first map only induces an isomorphism of first crystalline cohomology of the special fibre (relative to W) and the second map only induces an isomorphism of first étale cohomology of the (geometric) generic fibre.

6.1. **Raynaud's theorem on prolongations.** Lastly we give a geometric proof of Raynaud's theorem [Ray74, Théorème 3.3.3] on uniqueness of prolongations of finite flat commutative group schemes over a mildly ramified $\mathcal{O}_K$.

Let G_K be a finite flat commutative group scheme over K. A prolongation of G_K is a finite flat commutative group scheme G over $\mathcal{O}_K$ together with an isomorphism of its generic fiber with G_K (as finite flat commutative group schemes). Once G_K is fixed, its prolongations form a category with homomorphisms given by maps of group schemes compatible with the isomorphisms of their generic fiber.

Recall [Ray74, Corollaire 2.2.3.] that the (possibly empty) category of prolongations of a finite flat group scheme G over K has an initial $G_{\min}$ and a terminal object $G_{\max}$. Moreover these two are interchanged under Cartier duality.

Theorem 6.17 (c.f. [Ray74, Théorème 3.3.3])**.** *Assume G_K is a finite flat commutative group scheme which has a prolongation over $\mathcal{O}_K$.*

(1) If $e < p - 1$, then the prolongation is unique;

[12] To quote Sir Humphrey Appleby: "It is not for a humble mortal such as I to speculate on the complex and elevated deliberations of the mighty." But we suspect this is what Breuil had in mind when he claimed that his conjecture holds for H^1.

(2) If $e < 2(p-1)$, then the reduction of the canonical map $G_{\min} \to G_{\max}$ has kernel and cokernel annihilated by p;

(3) If $e = p-1$, then the reduction of the above map has étale kernel and multiplicative type cokernel.

Proof. To ease the notation, let us denote $G_1 \coloneqq G_{\min}$ and $G_2 \coloneqq G_{\max}$. Denote the canonical map by $\rho\colon G_1 \to G_2$. Choose a group scheme embedding $G_2 \to \mathcal{A}$ of G_2 into an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$, which is guaranteed by yet another Theorem of Raynaud (see [BBM82, Théorème 3.1.1]).

We shall contemplate with the quotient stack $[\mathcal{A}/G_1]$, which is a smooth proper Artin stack. Similar to Construction 6.11, let us pick a smooth complete-intersection $\mathcal{Y}$ with a fixed-point free action by G_1, let $\mathcal{Z} \coloneqq (\mathcal{A} \times_{\mathcal{O}_K} \mathcal{Y})/G_1$, which is a smooth projective scheme over $\mathcal{O}_K$, thanks to the second factor. Moreover $\mathcal{Z}$ is pointed because it admits a map from $\mathcal{A}$, which has a canonical point given by the identity section.

Let H be the image group scheme of the map $\rho_k\colon G_{1,k} \to G_{2,k}$. Applying the same reasoning as in Remark 6.10 shows us that the canonical map $f\colon \mathrm{Alb}(\mathcal{Z}_0) \to \mathrm{Alb}(\mathcal{Z})_0$ is identified with $\mathcal{A}_0/H \to \mathcal{A}_0/G_{2,k}$, whose kernel group scheme is given by $G_{2,k}/H$, which is none other than the cokernel of ρ_0. Now our statements on $\mathrm{coker}(\rho_0)$ follows directly from applying Corollary 4.6 to our $\mathcal{Z}$. The statements on $\ker(\rho_0)$ follows from Cartier duality. $\square$

Remark 6.18. Note that Raynaud first proved his theorem on prolongations, then use it to prove statements concerning Picard scheme of a p-adic integral scheme, which is directly related to statements concerning natural map between Albanese of reduction and reduction of Albanese, see Remark 4.8. Our roadmap is the exact opposite.

Finally, we remark that the estimate of s so that p^s kills the corkernel of $G_{\min} \to G_{\max}$ has been studied before, see for example, [VZ12] and [Bon06] (and the references therein), which used completely different methods than ours. Note that an affirmative answer to our Question 3.7 for $i = 2$, when specialized to the construction made in the above proof, agrees with Bondako's sharp estimate.

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