INTEGRAL p-ADIC HODGE FILTRATIONS IN LOW DIMENSION AND RAMIFICATION

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ABSTRACT. Given an integral p-adic variety, we observe that if the integral Hodge–de Rham spectral sequence behaves nicely, then the special fiber knows the Hodge numbers of the generic fiber. Applying recent advancements of integral p-adic Hodge theory, we show that such a nice behavior is guaranteed if the p-adic variety can be lifted to an analogue of second Witt vectors and satisfies some bound on dimension and ramification index. This is a (ramified) mixed characteristic analogue of results due to Deligne–Illusie and Fontaine–Messing. Lastly, we discuss an example illustrating the necessity of the aforementioned lifting condition, which is of independent interest.

CONTENTS

1. INTRODUCTION

Given a smooth proper scheme $\mathcal{X}$ over some p-adic ring of integers $\mathcal{O}_K$, can we tell the Hodge diamond of its generic fiber X by simply staring at the geometry of the special fiber $\mathcal{X}_0$? In general, there is no hope of this being true. But surely if one puts some constraints, this will be true.

There are two typical pathological phenomenons concerning Hodge and de Rham cohomology groups in an integral p-adic situation: one being torsions in the cohomology groups, the other being the non-degeneracy of the integral Hodge–de Rham spectral sequence. In this paper we convey the idea that, for the question asked in the beginning, the trouble comes from the second phenomenon. To be more precise, we define virtual Hodge numbers for any smooth proper variety in characteristic

p, see Definition 3.1. In Proposition 3.4, we observe that if the integral Hodge–de Rham spectral sequence of a lift $\mathcal{X}$ degenerates saturatedly (see Definition 2.3), then the virtual Hodge numbers of $\mathcal{X}_0$ agree with the Hodge numbers of the generic fiber X.

Then we show the following:

Theorem 1.1 (Main Theorem). *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be a smooth proper formal scheme, with $\mathfrak{S}$ being the Breuil–Kisin prism of $\mathcal{O}_K$ and E an associated Eisenstein polynomial. Assume that*

(1) there is a lift of $\mathcal{X}$ over $\mathfrak{S}/(E^2)$; and
(2) the relative diemension of $\mathcal{X}$ and the ramification index e of $\mathcal{O}_K$ satisfy inequality: $\dim X \cdot e < p - 1$.

Then the Hodge–de Rham spectral sequence for $\mathcal{X}$ is split degenerate. In particular, we have equality of (virtual) Hodge numbers:

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) = h^{i,j}(X).$$

One may think of this result as a mixed characteristic analogue of a theorem by Deligne–Illusie [DI87]. There is a similar statement with weaker conclusion when $\dim X$ exceeds the bound in (2), see Porism 3.10. In the case when $\mathcal{O}_K$ is the Witt ring of a perfect field, condition (1) is automatic and our result can also be deduced from Fontaine–Messing's result [FM87]. We summarize the implication relations between various relevant conditions in Section 3.4. The proof of the main theorem uses recent theory of prismatic cohomology due to Bhatt–Scholze [BS19] along with a result of Min [Min19]. For more details, see Section 3.

Lastly, one may wonder if either condition (1) or (2) is really necessary. Previously in [Li18] we have constructed a pair of relative 3-folds over $\mathbb{Z}_p[\zeta_p]$ with isomorphic special fiber, such that their generic fibers have different Hodge numbers, showing relevance of condition (2). In Section 4, which is independent from other sections, we discuss extensively an example illustrating the necessity of condition (1).

Theorem 1.2 (see Theorem 4.13). *There exists a smooth projective relative 4-fold $\mathcal{X}$ over a ramified degree two extension $\mathcal{O}_K$ of $\mathbb{Z}_p$, such that both of its Hodge–de Rham and Hodge–Tate spectral sequences are non-degenerate. Moreover the Hodge/conjugate filtrations are non-split as $\mathcal{O}_K$-modules.*

The idea of construction is as follows. The exotic group scheme α_p admits liftings over ramified p-adic rings of integers. We choose a lift G over a degree 2 ramified p-adic ring of integers, then we study the Hodge–de Rham and Hodge–Tate spectral sequences of BG, the classifying stack of G. With the aid of various computations in [ABM19], we find out that both are non-degenerate starting at degree 3 with non-split Hodge/conjugate filtrations starting at degree 2. In the end, we take an approximation of BG to get the desired example.

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2. Preliminaries on spectral sequences over DVRs

This section is a general discussion of spectral sequences associated with a filtered bounded perfect complex over a DVR.

Notation 2.1. Throughout this section, let R be a DVR with a uniformizer π. Denote $K \coloneqq R[1/\pi]$ and $\kappa \coloneqq R/\pi$. Let $(C, \mathrm{Fil}^\bullet)$ be a filtered object in $D^b_{\mathrm{Coh}}(R)$. We assume the filtration on C to be exhaustive and saturated.

Given a finitely generated R-module M, we denote by M_{tor} the torsion submodule in M, and we denote its torsion-free quotient by $M_{\mathrm{tf}} \coloneqq M/M_{\mathrm{tor}}$.

Remark 2.2. We do not assume this filtration to be either increasing or decreasing, as it is modeling both Hodge–de Rham and Hodge–Tate spectral sequences.

From $(C, \mathrm{Fil}^\bullet)$ we naturally get a spectral sequence converging from $H^i(\mathrm{Gr}^j)$ to $H^i(C)$. From now on, we will call it "the spectral sequence" if no confusion seems to arise. In the following definition, we refine the classical notion of the spectral sequence being degenerate.

Definition 2.3.

(1) We say the spectral sequence *degenerates* or *is degenerate* if for all pairs of integers (i, j), the natural map $H^i(\mathrm{Fil}^j) \to H^i(C)$ is an injection;
(2) We say the spectral sequence *degenerates saturatedly* or *is saturated degenerate* if it degenerates and the induced injection $H^i(\mathrm{Fil}^j)_{\mathrm{tf}} \to H^i(C)_{\mathrm{tf}}$ is saturated; and
(3) We say the spectral sequence *degenerates splittingly* or *is split degenerate* if it degenerates and the induced injection $H^i(\mathrm{Fil}^j) \to H^i(C)$ splits.

Recall that an injection/inclusion of torsion-free R-modules $N \subset M$ is said to be *saturated* if we have $\pi N = N \cap \pi M$ or, what is the same, the quotient M/N is π-torsion-free.

Remark 2.4. It is obvious that the spectral sequence being split degenerate implies it being saturated degenerate, and both implies it is degenerate.

In the case of Hodge–de Rham or Hodge–Tate spectral sequences (over mixed characteristic DVRs), we know that they degenerate after inverting p (see [Sch13, Corollary 1.8] and [BMS18, Theorem 1.7]). In this scenario, we have a condition on the infinite-page of the spectral sequence characterizing the spectral sequence being saturated or split degenerate:

Proposition 2.5. *Suppose that the spectral sequence degenerates after inverting π. Then*

(1) the spectral sequence is saturated degenerate if and only if

$$\mathrm{length}(H^i(C)_{\mathrm{tor}}) = \sum_j \mathrm{length}(H^i(\mathrm{Gr}^j C)_{\mathrm{tor}})$$

for all n; and

(2) the spectral sequence is split degenerate if and only if there is an abstract isomorphism of R-modules:

$$H^i(C) \simeq \bigoplus_j H^i(\mathrm{Gr}^j C).$$

Note that we assumed the $\mathrm{Fil}^\bullet$ to be exhaustive and saturated, the summation process is finite.

Proof of Proposition 2.5(1). First notice that

$$\mathrm{length}(H^i(C)_{\mathrm{tor}}) \leq \sum_j \mathrm{length}(\mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}) \leq \sum_j \mathrm{length}(H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}),$$

where the second inequality comes from the fact that the spectral sequence degenerates after inverting π (so $\mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}$ must be a subquotient of $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$). Therefore the equality condition implies equality between $\mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}$ and $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$. In other words, every element in $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$ is a permanent cycle. Since the spectral sequence degenerates after inverting π, we know all the differentials in the spectral sequence are torsion. Combining these two, we see that all the differentials are forced to be zero, which exactly means that the spectral sequence must degenerate.

Now we have reduced the statement of (1) to: assume the spectral sequence degenerates, then it is saturated degenerate if and only if the equality of lengths of π-torsions hold, and this statement is handled in the following Lemma 2.6.

Lemma 2.6. *Let M be a finitely generated R-module with an exhaustive and saturated filtration $\mathrm{F}^\bullet$. Then $\mathrm{F}^i_{\mathrm{tf}} \subset M_{\mathrm{tf}}$ is saturated for all i if and only if*

$$\mathrm{length}(M_{\mathrm{tor}}) = \sum_i \mathrm{length}(\mathrm{Gr}^i)_{\mathrm{tor}}).$$

Proof of the Lemma 2.6. First observe that for any i we have an inequality

$$\mathrm{length}(M_{\mathrm{tor}}) \leq \mathrm{length}(\mathrm{F}^i_{\mathrm{tor}}) + \mathrm{length}((M/\mathrm{F}^i)_{\mathrm{tor}}),$$

with equality holds if and only if the map

$$M_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}}$$

is surjective. Hence we see that the equality in condition is equivalent to

$$M_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}}$$

being surjective for all i.

Applying the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{F}^i_{\mathrm{tor}} \arrow[r] \arrow[d] & \mathrm{F}^i \arrow[r] \arrow[d] & \mathrm{F}^i_{\mathrm{tf}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_{\mathrm{tor}} \arrow[r] & M \arrow[r] & M_{\mathrm{tf}} \arrow[r] & 0 \end{tikzcd}$$

yields an exact sequence

$$0 \to M_{\mathrm{tor}}/\mathrm{F}^i_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}} \to (M_{\mathrm{tf}}/\mathrm{F}^i_{\mathrm{tf}})_{\mathrm{tor}} \to 0,$$

here we used the fact that $\mathrm{F}^i \cap M_{\mathrm{tor}} = \mathrm{F}^i_{\mathrm{tor}}$. This short exact sequence says exactly that the surjectivity of

$$M_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}}$$

is equivalent to $M_{\mathrm{tf}}/\mathrm{F}^i_{\mathrm{tf}}$ being torsion-free, hence concludes the proof of this lemma. □

□

Before proving the second part of Proposition 2.5, let us briefly discuss the condition of an extension of finitely generated torsion R-modules being split.

Definition 2.7. Let M be a finitely generated torsion R-module. Write

$$M = \oplus_i R/\pi^{n_i}$$

where $n_1 \leq n_2 \leq \ldots \leq n_l$. Then the *characteristic polygon* of M, denoted by $\mathcal{P}_M$, is the graph of the piece-wise linear function defined on $[0, l]$ passing through $(0, 0)$ with the slope of the i-th segment being n_i.

Remark 2.8. It is easy to see that the width of $\mathcal{P}_M$ is given by $\dim_\kappa(M/\pi M)$, and the end points are given by $(0, 0)$ and $(\dim_\kappa(M/\pi M), \mathrm{length}(M))$.

Given two finitely generated torsion R-modules, we would like to compare the characteristic polygons of an extension class and that of their direct sums.

Example 2.9 (see also [dJ93, P. 502])**.** Consider an extension:

$$0 \to N = R/\pi^l \to M \to R/\pi^m \to 0,$$

then we must have either $M \simeq R/\pi^{l+m}$, or $M \simeq R/\pi^n \oplus R/\pi^{m+l-n}$ where $\min\{n, m+l-n\} \leq \min\{l, m\}$ with equality if and only if the short exact sequence splits.

We observe the former case corresponds to $N/\pi \to M/\pi$ being not injective. In the latter case we see that $\mathcal{P}_M$ is always lower than or equal to $\mathcal{P}_{N \oplus M/N}$, and equality holds exactly when the extension class splits.

Inspired by this example, we give the following criterion characterizing split short exact sequences of finitely generated torsion R-modules.

Proposition 2.10. *Let M be a finitely generated torsion R-module, and $N \subset M$ is a submodule. Suppose $N/\pi \to M/\pi$ is an injection. Then $\mathcal{P}_M$ is lower than or equal to $\mathcal{P}_{N \oplus M/N}$, with equality holds if and only if $N \subset M$ splits.*

Proof. First assume we can prove the statement when N is cyclic (generated by one element). Write $N = N_1 \oplus N_2$, inducting on the dimension of N/π, we see that

$$\mathcal{P}_M \leq \mathcal{P}_{N_1 \oplus M/N_1} \leq \mathcal{P}_{N_1 \oplus N_2 \oplus M/N} = \mathcal{P}_{N \oplus M/N}$$

with equality holds if and only if both $N_1 \subset M$ and $N_2 \subset M/N_1$ split, or equivalently $N = N_1 \oplus N_2 \subset M$ splits. Therefore we reduce to the case where $N = R/\pi^n$ is cyclic.

Dually, we may induct on the dimension of $(M/N)/\pi$. By the same argument as above, we may also assume that M/N is also cyclic. Now we have reduced the statement to the case where both of N and M/N are cyclic, which is discussed in Example 2.9. $\square$

We may extend this discussion to a multi-filtered situation, which is useful in considerations of spectral sequences. Below let us record one consequence of Proposition 2.10.

Corollary 2.11. *Let $(M, \mathrm{F}^\bullet)$ be a finitely generated torsion R-module with an exhaustive and saturated filtration. Suppose that we have an abstract isomorphism*

$$M \simeq \bigoplus_i \mathrm{Gr}^i,$$

then all of $\mathrm{F}^i \subset M$ are direct summands.

Proof. Without loss of generality, let us assume that the filtration is increasing and $\mathrm{F}^i = 0$ for $i < 0$. Now Proposition 2.10 implies that

$$\mathcal{P}_M \leq \mathcal{P}_{\mathrm{Gr}^0 \oplus M/\mathrm{F}^0} \leq \mathcal{P}_{\mathrm{Gr}^0 \oplus \mathrm{Gr}^1 \oplus M/\mathrm{F}^1} \leq \ldots \leq \mathcal{P}_{\bigoplus_i \mathrm{Gr}^i},$$

and our condition forces all the inequalities above to be an equality. Hence applying Proposition 2.10 again yields what we want. ☐

Now we turn to the proof of (the "if" part of) Proposition 2.5(2).

Proof of Proposition 2.5(2). By validity of (1), we see that in this situation, the spectral sequence is already saturated degenerate. Therefore it suffices to show that the induced filtration $H^i(\mathrm{Fil}^j C)_{\mathrm{tor}} = \mathrm{Fil}^j H^i(C)_{\mathrm{tor}}$ on $H^i(C)_{\mathrm{tor}}$ is split for all i.

Notice that in our proof of (1), we established that the graded pieces of this filtration is exactly given by $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$. Now our condition implies that we have an abstract isomorphism

$$H^i(C)_{\mathrm{tor}} \simeq \bigoplus_j \mathrm{Gr}^j H^i(C)_{\mathrm{tor}}.$$

Applying Corollary 2.11, we see that $H^i(\mathrm{Fil}^j C)_{\mathrm{tor}} = \mathrm{Fil}^j H^i(C)_{\mathrm{tor}} \subset H^i(C)_{\mathrm{tor}}$ is split for all i. ☐

The following Proposition 2.12 will be used in later sections.

Proposition 2.12. *Let M be a finitely generated R-module, with $N \subset M$ a submodule. Suppose that $N_{\mathrm{tf}} \subset M_{\mathrm{tf}}$ is saturated. Then we have equality of dimensions:*

$$\dim_K N[1/\pi] = \dim_\kappa \mathrm{Im}(N/\pi \to M_{\mathrm{tf}}/\pi).$$

Proof. Both sides do not change when we passing from M to M_{tf}. Hence we may assume M to be torsion-free, in which case N is a direct summand by being a saturated submodule, and the statement becomes trivial in this situation. ☐

3. Consequences of recent developments in integral p-adic Hodge theory

In this section we concentrate ourselves in the p-adic situation.

3.1. **Notations and Setup.** Throughout this section, let K be a complete p-adic field with a chosen uniformizer π, ring of integers $\mathcal{O}_K$ and perfect residue field $\kappa \coloneqq \mathcal{O}_K/(\pi)$. Recall that $\mathcal{O}_K$ contains the ring of Witt vectors of κ, and the degree of their fraction field extension $K_0 \coloneqq \mathrm{W}(\kappa)[1/p] \subset K$ is called ramification index of K and denoted by e. This paper concerns low ramification situation, in particular we assume that $e \leq p-1$, therefore the ideal $(\pi) \subset \mathcal{O}_K$ has a unique divided power structure.

Denote $\mathfrak{S} \coloneqq \mathrm{W}(\kappa)[\![u]\!]$ with a surjection $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$, where u is sent to the chosen uniformizer π. The kernel of this surjection is generated by an Eisenstein polynomial $I = (E(u))$. Define the Frobenius $\phi\colon \mathfrak{S} \to \mathfrak{S}$ that extends the Frobenius on $\mathrm{W}(\kappa)$ and sends u to u^p, since $\mathfrak{S}$ is p-torsion free, this puts a unique δ-structure on $\mathfrak{S}$. The pair $(\mathfrak{S}, I)$ is a Breuil–Kisin type prism (see [BS19, Example 1.3 (3)]).

Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathcal{O}_K$. We denote the special fiber of $\mathcal{X}$ by $\mathcal{X}_0 \coloneqq \mathcal{X} \times_{\mathcal{O}_K} \kappa$ and the (rigid) generic fiber of $\mathcal{X}$ by $X \coloneqq \mathcal{X} \times_{\mathcal{O}_K} K$. We call $\mathcal{X}$ a *lifting of $\mathcal{X}_0$ over $\mathcal{O}_K$*. In the case where $e = 1$, i.e. $\mathcal{O}_K = \mathrm{W}(\kappa)$, we call $\mathcal{X}$ an *unramified lift of $\mathcal{X}_0$*.

3.2. **Virtual Hodge Numbers.** Recall that we have a natural identification $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa)) \otimes^{\mathbb{L}}_{\mathrm{W}(\kappa)} \kappa \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}_0/\kappa)$, which implies that we have a natural injection

$$H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \hookrightarrow H^i_{\mathrm{dR}}(\mathcal{X}_0/\kappa),$$

for all i. Therefore we may regard $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p$ as a natural subquotient of $H^i_{\mathrm{dR}}(\mathcal{X}_0/\kappa)$.

Definition 3.1 (virtual Hodge numbers)**.** The Hodge filtrations on $H^i_{\mathrm{dR}}(\mathcal{X}_0/\kappa)$ induces natural filtrations on the subquotient $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p$. The virtual Hodge numbers of $\mathcal{X}_0$, is given by

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) := \dim_\kappa \mathrm{Fil}^i(H^{i+j}_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p) - \dim_\kappa \mathrm{Fil}^{i+1}(H^{i+j}_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p)$$

Unwinding the definition, we have the following description of the i-th induced filtration on $H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p$:

($\boxdot$)

$$\mathrm{Im}(\mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}_0/\kappa}) \to H^n(\Omega^\bullet_{\mathcal{X}_0/\kappa})) \bigcap H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \to H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p).$$

Note that this definition *only depends* on the smooth proper variety $\mathcal{X}_0$ in characteristic p.

Remark 3.2. It is worth pointing out that in this definition, we may replace the Witt vectors by any ring of integers $\mathcal{O}_K$ as long as the ramification index $e \leq p-1$. By the virtue of base change formula of crystalline cohomology [BO78, Theorem 7.8], we have natural identifications:

- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa)) \otimes_{\mathrm{W}(\kappa)} \mathcal{O}_K \cong H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)$,
- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}} \otimes_{\mathrm{W}(\kappa)} \mathcal{O}_K \cong H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}$,
- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \cong H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)/\pi$, and
- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p \cong H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi$.

These filtrations are only objects in characteristic p. The goal of this subsection is to show that if the integral p-adic Hodge filtration behaves nicely, then these filtrations can tell us something about the rational Hodge filtrations.

Proposition 3.3. *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be as in Subsection 3.1. Assume that the integral Hodge–de Rham spectral sequence of $\mathcal{X}$ degenerates, then we have identifications of subspaces in $H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})/\pi \cong H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p$:*

$$\begin{array}{c} \mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})/\pi) \\ \Big\downarrow \cong \\ \mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}_0/\kappa}) \to H^n(\Omega^\bullet_{\mathcal{X}_0/\kappa})) \bigcap H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \end{array}$$

for all i.

Proof. The assumption on Hodge–de Rham spectral sequence yields a short exact sequence

$$0 \to H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K}) \to H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K}) \to H^n(\Omega^{\leq i-1}_{\mathcal{X}/\mathcal{O}_K}) \to 0.$$

Modulo π, we get

$$H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^{\leq i-1}_{\mathcal{X}/\mathcal{O}_K})/\pi \to 0.$$

Now we have the following diagram with horizontal lines being exact and vertical arrows being injective:

$$\begin{tikzcd}
H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})/\pi \arrow[r] \arrow[d, hook] & H^n(\Omega^{\bullet}_{\mathcal{X}/\mathcal{O}_K})/\pi \arrow[r] \arrow[d, hook] & H^n(\Omega^{\leq i-1}_{\mathcal{X}/\mathcal{O}_K})/\pi \arrow[d, hook] \\
H^n(\Omega^{\geq i}_{\mathcal{X}_0/\kappa}) \arrow[r] & H^n(\Omega^{\bullet}_{\mathcal{X}_0/\kappa}) \arrow[r] & H^n(\Omega^{\leq i-1}_{\mathcal{X}_0/\kappa}).
\end{tikzcd}$$

A simple diagram chasing, together with the identification spelled out in Remark 3.2, yields the claimed identification. ◻

Recall the definition of a spectral sequence being saturated degenerate in Definition 2.3.

Proposition 3.4. *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be as in Subsection 3.1. Assume that the integral Hodge–de Rham spectral sequence of $\mathcal{X}$ is saturated degenerate, then we have equality of (virtual) Hodge numbers:*

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) = h^{i,j}(X).$$

Proof. According to the definitions, we need to show that the dimension of i-th filtration on $H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p \cong H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi$ agrees with the dimension of i-th filtration on $H^n_{\mathrm{dR}}(X)$.

By Proposition 3.3, we may rewrite the formula $\boxdot$ of the i-th filtration on $H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi$ by

$$\mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^{\bullet}_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^{\bullet}_{\mathcal{X}/\mathcal{O}_K})_{\mathrm{tf}}/\pi \cong H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi).$$

Our assumption implies that the submodule $H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K}) \subset H^n(\Omega^{\bullet}_{\mathcal{X}/\mathcal{O}_K})$ meets the condition of Proposition 2.12. Hence we may apply Proposition 2.12 which gives us the claimed equality of dimensions of filtrations. ◻

3.3. **Main Theorem.** In this subsection, we explain the proof of the Main Theorem 1.1, which we repeat below:

Theorem 3.5. *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be a smooth proper formal scheme. Assume that*

(1) there is a lift of $\mathcal{X}$ over $\mathfrak{S}/(E^2)$; and

(2) the relative diemension of $\mathcal{X}$ and the ramification index satisfy inequality: $\dim X \cdot e < p - 1$.

Then the Hodge–de Rham spectral sequence for $\mathcal{X}$ is split degenerate. In particular, we have equality of (virtual) Hodge numbers:

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) = h^{i,j}(X).$$

Remark 3.6.

(1) After modulo (u), the surjection $\mathfrak{S}/(E^2) \twoheadrightarrow \mathcal{O}_K$ becomes $\mathrm{W}_2(\kappa) \twoheadrightarrow \kappa$ (note that E is an Eisenstein polynomial in u). So we may view this Theorem as a mixed-characteristic analogue of a theorem by Deligne–Illusie [DI87, Corollaire 2.4].

(2) Together with the observation that the Euler characteristic of a flat coherent sheaf is locally constant in a flat family, our result may be strengthened to the following. A smooth proper variety $\mathcal{X}_0$ knows Hodge numbers of the

generic fiber of a (formal) lifting provided: (1) the lifting can be further lifted to $\mathfrak{S}/(E^2)$; and (2) we have inequality $(\dim \mathcal{X}_0 - 1) \cdot e < p - 1$.

(3) The condition (1) is not so easy to verify. There are two cases we can think of in which this condition is automatically guaranteed. The first being that $\mathcal{X}$ is an unramified lift (for then the surjection $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ admits a section), in which case a stronger statement follows from the work of Fontaine–Messing [FM87], see Remark 3.11. The second case is when $\mathcal{X}_0$ has unobstructed deformation theory, e.g. when $H^2(\mathcal{X}_0, \mathrm{T}) = 0$, by deformation theoretic considerations.

(4) Similar to the situation of Deligne–Illusie's statement, there are examples showing the necessity of condition (1). In fact one example comes from (ramified) liftings of (counter-)examples to Deligne–Illusie's statement in characteristic p due to Antieau–Bhatt–Mathew [ABM19], see Section 4 and more precisely Theorem 4.13.

The main ingredients that go into the proof are some recent developments in integral p-adic Hodge theory. So we shall postpone the proof until after the introduction of these ingredients.

Recently Bhatt–Scholze [BS19] developed prismatic cohomology theory to unite many (if not all) known p-adic cohomology theories one may attach to a p-adic smooth formal scheme over a certain class of p-adic base rings. While their theory is in a much broader context, we shall specialize their results to the situation of interest for this paper: they introduced prismatic site $(\mathcal{X}/\mathfrak{S})_{\Delta}$ on which there are two structure sheaves $\mathcal{O}_{\Delta}$ and $\overline{\mathcal{O}}_{\Delta} \coloneqq \mathcal{O}_{\Delta}/E$. The cohomology of $\mathcal{O}_{\Delta}$ on $(\mathcal{X}/\mathfrak{S})_{\Delta}$ is denoted by $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$. There is a natural map of ringed topoi (see [BS19, Construction 4.4]):

$$\nu \colon \mathrm{Shv}((\mathcal{X}/\mathfrak{S})_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \to \mathrm{Shv}(\mathcal{X}_{\text{ét}}, \mathcal{O}_{\mathcal{X}}).$$

The properties we need of these objects are summarized in the following:

Theorem 3.7 (Bhatt–Scholze)**.**

(1) (see [BS19, Corollary 15.4]*) There is a canonical isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \cong \mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \phi_* \mathcal{O}_K,$$

where $\phi_*\mathcal{O}_K$ is $\mathcal{O}_K$ viewed as an $\mathfrak{S}$-module (or even algebra) via the composite $\mathfrak{S} \xrightarrow{\phi} \mathfrak{S} \to \mathcal{O}_K = \mathfrak{S}/(E)$.

(2) (see [BS19, Theorem 4.10]*) There is a canonical isomorphism*

$$\Omega^i_{\mathcal{X}/\mathcal{O}_K}\{-i\} \cong R^i\nu_*\overline{\mathcal{O}}_{\Delta}.$$

Here $(-)\{j\} \coloneqq (-) \otimes_{\mathcal{O}_K} ((E)/(E^2))^{\otimes j}$. This induces an increasing filtration (called the conjugate filtration) on $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K$ giving rise to an E_2 spectral sequence (called the Hodge–Tate spectral sequence):

$$(\boxdot)\quad E_2^{i,j} = H^i(\mathcal{X}, \Omega^j_{\mathcal{X}/\mathcal{O}_K})\{-j\} \Longrightarrow H^{i+j}_{\mathrm{HT}}(\mathcal{X}/\mathcal{O}_K) \coloneqq H^{i+j}(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K).$$

(3) (see [BS19, Remark 4.13 and Proposition 4.14]*, c.f.* [ALB19, Proposition 3.2.1]*) The map*

$$\mathcal{O}_{\mathcal{X}} \to \tau^{\leq 1} R\nu_* \overline{\mathcal{O}}_{\Delta}$$

splits if and only if $\mathcal{X}$ lifts to $\mathfrak{S}/(E^2)$.

Lastly we need a result of Min [Min19] concerning the $\mathfrak{S}$-module structure of $H^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ in the case of small i and low ramification index:

Theorem 3.8 (Min [Min19, Theorem 5.8])**.** *When $i \cdot e < p-1$, we have an abstract isomorphism of $\mathfrak{S}$-modules:*

$$H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathfrak{S}^n \oplus \bigoplus_{j \in J} \mathfrak{S}/p^{n_j},$$

where J is a finite set.

Using the result of Min and our analysis of spectral sequences in the previous section, we may relate the behavior of the Hodge–de Rham and the Hodge–Tate spectral sequences.

Corollary 3.9. *Assume $\dim X \cdot e < p-1$, we have two equivalences:*

(1) The Hodge–de Rham spectral sequence being saturated degenerate is equivalent to the Hodge–Tate spectral sequence being saturated degenerate; and
(2) The Hodge–de Rham spectral sequence being split degenerate is equivalent to the Hodge–Tate spectral sequence being split degenerate.

Proof. Notice that $\mathfrak{S}$ and $\mathfrak{S}/p^\ell$ are, as $\mathfrak{S}$-modules, Tor-independent with $\mathcal{O}_K$ and $\phi_* \mathcal{O}_K$. Therefore in the situation considered, by Theorem 3.7 (1) and (2) and Theorem 3.8, we have an abstract isomorphism of $\mathcal{O}_K$-modules:

$$H^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \simeq H^i_{\mathrm{HT}}(\mathcal{X}/\mathcal{O}_K)$$

for all i.

Moreover we know that both spectral sequences degenerate after inverting π (see [Sch13, Corollary 1.8] and [BMS18, Theorem 1.7]). Hence we reduce these two statements respectively to Proposition 2.5 (1) and (2), by observing that the starting pages of these two spectral sequences are formed by abstractly isomorphic $\mathcal{O}_K$ modules (after switching bi-degrees). $\square$

Finally we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By anti-symmetrizing (c.f. [DI87, step (a) of Theorem 2.1]) the section

$$\Omega^1_{\mathcal{O}_{\mathcal{X}}}\{-1\}[-1] \to \tau^{\leq 1} R\nu_* \overline{\mathcal{O}}_{\Delta}$$

given by (see Theorem 3.7 (3)) the lift of $\mathcal{X}$ to $\mathfrak{S}/(E^2)$, we see that the conjugate filtration splits (note that our constraint on $\dim X$ in particular implies the relative dimension is smaller than p):

$$\bigoplus_{i=0}^{\dim X} \Omega^i_{\mathcal{O}_{\mathcal{X}}}\{-i\}[-i] \simeq R\nu_* \overline{\mathcal{O}}_{\Delta}.$$

Therefore we see that the Hodge–Tate spectral sequence ⚁ is split degenerate. By Corollary 3.9 we see that the Hodge–de Rham spectral sequence must also be split degenerate. The last statement concerning numerical equalities follows from Proposition 3.4. $\square$

In the situation where $\dim X$ exceeds the bound, our argument produces the following.

Porism 3.10. *Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathcal{O}_K$ which lifts to $\mathfrak{S}/(E^2)$, let us denote the relevant threshold by $T := \frac{p-1}{e}$. Then the differentials in the Hodge-de Rham spectral sequence with target of total degree $< T-1$ are zero, the induced Hodge filtrations on de Rham cohomology of degree $< T-1$ are saturated, and the induced Hodge filtrations is split for degrees $< T-2$. Hence we have $\mathfrak{h}^{i,j}(\mathcal{X}_0) = H^{i,j}(X)$, for $i+j+2 < T$ (or equivalently $(i+j+2)\cdot e < p-1$).*

Proof. The prismatic cohomology $H^a_{\Delta}(\mathcal{X}/\mathfrak{S})$ is isomorphic to the base change of the a-th étale cohomology (of the geometric rigid generic fiber), for $a < T$. Therefore the b-th Hodge–Tate and de Rham cohomology are abstractly isomorphic, for $b < T-1$. The Hodge–Tate decomposition still holds in this range, as $T-1 \leq p-1$. Therefore the length of torsions in $H^b_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ is equal to the sum of their Hodge counterparts, which implies the induced map $H^b(\mathcal{X}, \Omega^{\geq j})_{\mathrm{tor}} \to H^b_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)_{\mathrm{tor}}$ is injective, for $b < T-1$ and arbitrary j. This shows the statement about vanishing of differentials in the given range. The statement about Hodge filtrations being saturated/split follows from the same argument of the "if" part of Proposition 2.5. The statement about equality of numbers follows from the proof of Proposition 3.4. □

In the case when $e = 1$, namely $\mathcal{X}_0$ has an unramified lifting, the result of Fontaine–Messing gives something slightly more.

Remark 3.11. It is a result of Fontaine–Messing [FM87, Corollary 2.7] that given an unramified lift $\mathcal{X}$, the Hodge–de Rham spectral sequence degenerates up to degree $p-1$, namely all the differentials with *target* of total degree $\leq p-1$ are zero. Moreover they showed that the integral Hodge filtrations on $H^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$, where $0 \leq i \leq p-1$, are equipped with divided Frobenius structure and altogether these form so-called Fontaine–Laffaille modules [FL82]. In particular, by a result of Wintenberger [Win84], the Hodge filtrations are split submodules in the range $0 \leq i \leq p-1$ (c.f. [FM87, Remark 2.8.(ii)]). These results imply the following:

Corollary 3.12 (Corollary of [FM87, Corollary 2.7])**.** *Let $\mathcal{X}$ be an unramified lift of a smooth proper variety $\mathcal{X}_0$.*

(1) If $\dim \mathcal{X}_0 \leq p-1$, then the Hodge numbers of X are determined by $\mathcal{X}_0$.
(2) In general, the degree $\leq p-2$ Hodge numbers of X are determined by $\mathcal{X}_0$.

Comparing their results concerning unramified liftings with our result, we see that our approach (specialized to unramified liftings) so far can only prove analogous facts with $\mathcal{X}_0$ of 2 dimension less. This is due to:

(1) Min's result was established by proving the i-th prismatic cohomology of $\mathcal{X}/\mathfrak{S}$ shares the same structure as the i-th étale cohomology of the (geometric) generic fiber X, which only holds when $i \cdot e < p-1$; and
(2) in our general argument we have no control of the contribution of E-torsion of prismatic cohomology groups beyond the designated range.

Perhaps both of these two obstacles may be overcome in the unramified case, which would recover Fontaine–Messing result in this particular direction.

3.4. **Summary.** Given $\mathcal{X}/\mathcal{O}_K$ as in Section 3.1, there are the following conditions on $\mathcal{X}/\mathcal{O}_K$:

C.1 The formal scheme $\mathcal{X}$ lifts to $\mathfrak{S}/(E^2)$;
C.2 The Hodge–Tate spectral sequence degenerates splittingly;
C.3 The Hodge–Tate spectral sequence degenerates saturatedly;

C.4 The Hodge–de Rham spectral sequence degenerates splittingly;
C.5 The Hodge–de Rham spectral sequence degenerates saturatedly;
C.6 The virtual Hodge numbers of $\mathcal{X}_0$ is equal to the Hodge numbers of X.

The relations between these conditions are summarized in the following diagram:

$$\begin{tikzcd}
\mathbf{C.1} \arrow[r, Rightarrow, "\alpha"] & \mathbf{C.2} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\beta"] & \mathbf{C.3} \arrow[d, Leftrightarrow, "\beta"] \\
 & \mathbf{C.4} \arrow[r, Rightarrow] & \mathbf{C.5} \arrow[d, Rightarrow, "\gamma"] \\
e = 1 \arrow[uu, Rightarrow] \arrow[urr, Rightarrow, "\delta"] & & \mathbf{C.6}.
\end{tikzcd}$$

Below we remind readers under what condition (and why) we have some of these implications:

- α holds when $\dim X \le p-1$ and follows from [BS19, Remark 4.13 and Proposition 4.14];
- β holds when $\dim X \cdot e < p-1$ and follows from Min's work [Min19] together with the analysis of relevant spectral sequences, see Corollary 3.9;
- γ always holds and is the content of Proposition 3.3; and
- δ holds provided $\dim X \le p-1$ and follows from the work of Fontaine–Messing [FM87, Corollary 2.7].

4. Lifting the example of Antieau–Bhatt–Mathew

One might wonder if it is really necessary to have both of conditions (1) and (2) in Theorem 1.1 in order for the Hodge–de Rham spectral sequence to behave nicely. We would like to mention that in [Li18] we found pairs of relatively 3-dimensional smooth projective schemes over $\mathbb{Z}_p[\zeta_p]$, such that their special fibers are isomorphic but the degree 2 Hodge numbers of their generic fibers are different. Therefore these give rise to examples where the Hodge–de Rham spectral sequence is not saturated degenerate by Proposition 3.4. The cohomological degree times ramification index of these examples are twice of $p-1$, illustrating necessity of having a condition like (2) in Theorem 1.1.

In this last section we would like to illustrate by an example, of the necessity of condition (1) in Theorem 1.1. More precisely, we shall construct smooth proper schemes over degree 2 ramified extensions of $\mathbb{Z}_p$, such that the Hodge–de Rham spectral sequence are not degenerate (starting at degree 3), and the Hodge filtrations are non-saturated (starting at degree 2). The idea is to approximate the classifying stack of a lift of α_p (which only exists over a ramified ring of integers), and the key computations and techniques are already in [ABM19].

4.1. **Recollection of** [TO70]**.** In this subsection, we give a preliminary discussion of group schemes of order p over p-adic base rings. Fix a scheme S over $\mathbb{Z}_p$. Recall that in [TO70], the authors made a detailed study of finite flat group schemes of order p over S, below let us summarize their results.

Firstly, all such group schemes are commutative [TO70, Theorem 1]. Secondly, for each p there is a unit $\omega \in \mathbb{Z}_p^*$ (which is denoted as ω_{p-1} in loc. cit.) and a bijection [TO70, Theorem 2] between

(1) isomorphism classes of finite flat order p group schemes over S; and

(2) isomorphism classes of triples $(\mathcal{L}, a, b)$ where $a \in \Gamma(S, \mathcal{L}^{\otimes(p-1)})$ and $b \in \Gamma(S, \mathcal{L}^{\otimes(1-p)})$, and they satisfy the relation $a \otimes b = p\omega$,

here we have identified $\mathcal{L}^{\otimes(p-1)} \otimes \mathcal{L}^{\otimes(1-p)} \cong \mathcal{O}_S$. The group associated with $(\mathcal{L}, a, b)$ is denoted by $\mathrm{G}^{\mathcal{L}}_{a,b}$, with underlying scheme structure given by $\underline{\mathrm{Spec}}(\mathcal{O}_S \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{-p+1})$ where the ring structure comes from $\mathcal{L}^{-p} \xrightarrow{a} \mathcal{L}^{-1}$ [TO70, P. 12]. So $\mathrm{G}^{\mathcal{L}}_{a,b}$ is an étale group scheme if and only if $a \in \Gamma(S, \mathcal{L}^{\otimes(p-1)})$ is an invertible section [TO70, P. 16 Remark 6]. Moreover the Cartier dual of $\mathrm{G}^{\mathcal{L}}_{a,b}$ is[1] given by $\mathrm{G}^{\mathcal{L}^{-1}}_{b,a}$ [TO70, P. 15 Remark 2].

Example 4.1. When $S = \mathrm{Spec}(\mathbb{F}_p)$, there is only one line bundle on S, namely $\mathcal{O}_S$. Furthermore we have $p = 0$ on S. Hence we see that group schemes of order p over S are classified by pairs $(a, b) \in \mathbb{F}_p^2$ with the constraint that $ab = 0$. Note that these pairs have no nontrivial automorphism as any invertible element $u \in \mathbb{F}_p^*$ satisfies $u^{p-1} = 1$. There are three possibilities:

(1) $a \neq 0$, which forces $b = 0$, corresponding to a form of the étale group scheme $\mathbb{Z}/p$. When $a = 1$, it is $\mathbb{Z}/p$.
(2) Dually we can have $b \neq 0$ and $a = 0$, corresponding to a form of μ_p. It is μ_p when $b = 1$.
(3) Lastly if both of $a = b = 0$, we get α_p.

4.2. **A stacky example.** Now we specialize to the case where $S = \mathrm{Spec}(\mathcal{O}_K)$ is given by the valuation ring of a p-adic field. There is no non-trivial line bundle on a local scheme such as S. In order to lift α_p from the residue field of $\mathcal{O}_K$, it suffices to find an element $\pi \in \mathfrak{m}$ such that $p/\pi \in \mathfrak{m}$. Here $\mathfrak{m}$ denotes the maximal ideal in $\mathcal{O}_K$. We see that $\mathcal{O}_K$ cannot be absolutely unramified, and as long as it is ramified, we may find such an element π. From now on, let us fix such a choice of $\mathcal{O}_K$ and π.

Notation 4.2. Let K be a degree 2 ramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, a uniformizer π in the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$. Then $\pi' \coloneqq p\omega/\pi$ is a uniformizer as well. Denote $S \coloneqq \mathrm{Spec}(\mathcal{O}_K)$ and let $G \coloneqq G^{\mathcal{O}_S}_{\pi,\pi'}$ be the lift of α_p over S corresponding to (π, π').

In the following we shall study the Hodge–Tate and Hodge–de Rham spectral sequence of BG. Note that BG is a smooth proper stack over $\mathrm{Spec}(\mathcal{O}_K)$ with special fiber $BG \times_{\mathcal{O}_K} \mathbb{F}_p \cong B\alpha_p$. The following computation of Antieau–Bhatt–Mathew is very useful.

Proposition 4.3 (see [ABM19, Proposition 4.10])**.** *If $p > 2$, the Hodge cohomology group of $B\alpha_p$ is given by*

$$H^*(B\alpha_p, \wedge^* L_{B\alpha_p/\mathbb{F}_p}) \cong E(\alpha) \otimes P(\beta) \otimes E(s) \otimes P(u)$$

where $E(-)$ (resp. $P(-)$) denotes the exterior (resp. polynomial) algebra on the designated generator. Here $\alpha \in H^1(B\alpha_p, \mathcal{O})$, $\beta \in H^2(B\alpha_p, \mathcal{O})$, $s \in H^0(B\alpha_p, L_{B\alpha_p/\mathbb{F}_p})$ and $u \in H^1(B\alpha_p, L_{B\alpha_p/\mathbb{F}_p})$. For $p = 2$ we replace $E(\alpha) \otimes P(\beta)$ with $P(\alpha)$.

Lemma 4.4. *The cotangent complex of BG is $L_{BG/\mathcal{O}_K} \simeq \mathcal{O}/(\pi)[-1]$.*

Proof. Observe that the equation of the underlying scheme of G is given by $x^p - \pi x$, hence we know that $L_{G/\mathcal{O}_K} \simeq \mathcal{O}_G/(\pi)$. Therefore the underlying coLie complex of G

[1] Note that they are commutative group schemes by first sentence of this paragraph.

is also $\mathcal{O}_G/(\pi)$. As G is commutative, our statement follows from [Ill72, Proposition 4.4].[2] $\square$

Remark 4.5. In the proof of [ABM19, Proposition 4.10], the authors showed that the Postnikov tower $\mathcal{O}_{B\alpha_p} \to L_{B\alpha_p/\mathbb{F}_p} \xrightarrow{a} \mathcal{O}_{B\alpha_p}[-1]$ of the cotangent complex of $B\alpha_p$ splits: $L_{B\alpha_p/\mathbb{F}_p} \simeq \mathcal{O}_{B\alpha_p} \oplus \mathcal{O}_{B\alpha_p}[-1]$. In our situation, we get a triangle in $D(BG)$:

$$\mathcal{O}[-1] \to L_{BG/\mathcal{O}_K} \to \mathcal{O}$$

where the connecting morphism is multiplication by π. Specializing to the special fiber $B\alpha_p$, we get a diagram

(⚂)
$$\begin{tikzcd}
\mathcal{O}[-1] \arrow[r] \arrow[d] & L_{BG/\mathcal{O}_K} \arrow[r] \arrow[dl, "b"'] \arrow[d] \arrow[dr, "c"] & \mathcal{O} \arrow[d] \\
\mathcal{O}/(\pi) \cdot u[-1] \arrow[r] & L_{B\alpha_p/\mathbb{F}_p} \arrow[r] \arrow[l, bend left, "a"] & \mathcal{O}/(\pi) \cdot s
\end{tikzcd}$$

where b is the identification $L_{BG/\mathcal{O}_K} \simeq \mathcal{O}/(\pi)[-1]$. This gives a particular choice of the splitting of $L_{B\alpha_p/\mathbb{F}_p} \simeq \mathcal{O}_{B\alpha_p} s \oplus \mathcal{O}_{B\alpha_p} u[-1]$, where the classes s and u are as in the statement of the aforementioned Proposition 4.3. Here let us name the map $sp\colon L_{BG/\mathcal{O}_K} \xrightarrow{b\oplus c} \mathcal{O}_{B\alpha_p} s \oplus \mathcal{O}_{B\alpha_p} u[-1] \simeq L_{B\alpha_p/\mathbb{F}_p}$ for we will use it later.

Next let us compute the Hodge cohomology groups of BG and identify the algebra structure.

Proposition 4.6. *For any pair of integers (i,j), we have*

$$H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) = \begin{cases} \mathcal{O}_K & i = j = 0 \\ \mathbb{F}_p & j = 0, i = 2m > 0 \text{ or } 0 < j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore specialization maps give rise to injections $sp\colon H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \hookrightarrow H^i(B\alpha_p, \wedge^j L_{B\alpha_p/\mathbb{F}_p})$ whenever $i + j > 0$. Moreover these injections are compatible with multiplication and differentials, and gives an identification

$$H^*(BG, \wedge^* L_{BG/\mathcal{O}_K}) = \begin{cases} (\mathcal{O}_K[\beta, u] \otimes E(\tau))/(\pi\tau, \pi\beta, \pi u, \tau^2) & p > 2 \\ (\mathcal{O}_K[\beta, u, \tau])/(\pi\tau, \pi\beta, \pi u, \tau^2 - \beta u^2) & p = 2, \end{cases}$$

where $\beta \in H^2(BG, \mathcal{O})$ and $u \in H^1(BG, L_{BG/\mathcal{O}_K})$ both specialize to the designated elements in the Hodge ring of $B\alpha_p$, and $\tau \in H^2(BG, L_{BG/\mathcal{O}_K})$ specializes to $\alpha u + \beta s$ (up to scale s by a unit).

Proof. First we begin with the computation of cohomology of $\mathcal{O}$. Similar to the first paragraph of proof of [ABM19, Proposition 4.10], we have $H^*(BG, \mathcal{O}) = \mathrm{Ext}^*_{\mathcal{O}_K[y]/(y^p - \pi' y)}(\mathcal{O}_K, \mathcal{O}_K)$ by Cartier duality. Here we used the fact that the Cartier dual of $G_{\pi,\pi'}$ is $G_{\pi',\pi}$ whose underlying scheme structure is $\mathrm{Spec}(\mathcal{O}_K[y]/(y^p - \pi' y))$ with its identity section given by $y = 0$. Using the standard resolution:

$$\left(\ldots \mathcal{O}_K[y]/(y^p - \pi' y) \xrightarrow{y^{p-1} - \pi'} \mathcal{O}_K[y]/(y^p - \pi' y) \xrightarrow{y} \mathcal{O}_K[y]/(y^p - \pi' y)\right) \simeq \mathcal{O}_K$$

one verifies the computation when $j = 0$.

[2] Note that the cotangent complex of BG is the coLie complex shifted by -1.

For the case when $j > 0$, just observe that we have

$$\wedge^* L_{BG/\mathcal{O}_K} = \wedge^* (\mathcal{O}/(\pi)[-1]) = \mathrm{Sym}^*(\mathcal{O}/(\pi))[-*].$$

Therefore we get

$$H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) = H^i(BG, \mathrm{Sym}^j \mathcal{O}/(\pi)[-j]) = H^{i-j}(B\alpha_p, \mathcal{O}),$$

which verifies the computation when $j > 0$ via Proposition 4.3.

The second statement follows from the fact that $H^i(BG, \wedge^j L_{BG/\mathcal{O}_K})$ are all π-torsion when $i+j > 0$ by the first sentence. In particular, by dimension consideration we see that the induced map $H^2(BG, \mathcal{O}) \to H^2(B\alpha_p, \mathcal{O})$ must be an isomorphism. Hence we may pick a generator $\beta \in H^2(BG, \mathcal{O})$ which lifts the designated generator in $H^2(B\alpha_p, \mathcal{O})$.

Now we deal with the statement concerning image of other specialization maps. Since the map b in ⚂ is an identification, we see that the b component of

$$H^*(BG, L_{BG/\mathcal{O}_K}) \xrightarrow{sp=b\oplus c} H^*(B\alpha_p, L_{B\alpha_p/\mathbb{F}_p}) = H^{*-1}(B\alpha_p, \mathcal{O}) \cdot u \oplus H^*(B\alpha_p, \mathcal{O}) \cdot s$$

is always an isomorphism. The map c factors through $L_{BG/\mathcal{O}_K} \to \mathcal{O} \to \mathcal{O}/(\pi) \cdot s$, using our computation of $H^*(BG, \mathcal{O})$, we see that the image of c component of sp is nonzero exactly on $H^{3+2*}(BG, L_{BG/\mathcal{O}_K})$. Put these together, one verifies the claim of specialization maps on $H^1(BG, L_{BG/\mathcal{O}_K})$ and $H^2(BG, L_{BG/\mathcal{O}_K})$.

For the last sentence, let us just prove the case when $p > 2$, the case of $p = 2$ can be proved in the same way. First we observe that we have

$$(\mathcal{O}_K[\beta, u] \otimes E(\tau)) \xrightarrow{f} H^*(BG, \wedge^* L_{BG/\mathcal{O}_K}) \xrightarrow{sp} H^*(B\alpha_p, \wedge^* L_{B\alpha_p/\mathbb{F}_p}),$$

with β, u and τ as in the statement. The map f must kill the relations $\pi\tau, \pi\beta, \pi u$, as the positive degree Hodge groups of BG are π-torsion. After quotient out the relations, we get an injection

$$(\mathcal{O}_K[\beta, u] \otimes E(\tau))/(\pi\tau, \pi\beta, \pi u, \tau^2) \xrightarrow{sp\circ f} H^*(B\alpha_p, \wedge^* L_{B\alpha_p/\mathbb{F}_p})$$

on positive degree part because of Proposition 4.3. Hence the map f induces injection

$$(\mathcal{O}_K[\beta, u] \otimes E(\alpha u))/(\pi\alpha u, \pi\beta, \pi u, (\alpha u)^2) \xrightarrow{f} H^*(BG, \wedge^* L_{BG/\mathcal{O}_K}).$$

By explicitly comparing dimensions of each bi-degree parts, one concludes that f must also be surjective. $\square$

Finally, we can understand the Hodge–de Rham spectral sequence of BG with the aid of [ABM19, Proposition 4.12].

Proposition 4.7. *In the Hodge–de Rham spectral sequence of BG, we have (up to unit) $d_1(\tau) = u^2$ and $d_1(\beta) = d_1(u) = 0$ for all p. The de Rham cohomology of BG is given by*

$$H^*_{\mathrm{dR}}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K[\beta']/(p\beta'),$$

where β' has degree 2.

Proof. The first sentence follows from the proof of [ABM19, Proposition 4.10] and the fact that specialization gives injection

$$sp\colon H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \hookrightarrow H^i(B\alpha_p, \wedge^j L_{B\alpha_p/\mathbb{F}_p})$$

which is compatible with multiplication and differentials.

Using the fact that d_1 is a differential, we see that on the E_2-page the non-zero entries are

$$E_2^{i,j} = \begin{cases} \mathcal{O}_K & i = j = 0 \\ \mathbb{F}_p \cdot \beta^n & i = 0, j = 2n > 0 \\ \mathbb{F}_p \cdot \beta^n u & i = 1, j = 2n+1 > 0. \end{cases}$$

In particular, there is no room for nonzero differentials, hence the spectral sequence degenerates on E_2 page. In particular, we see that the length of de Rham cohomology is as described in the statement of this Proposition. To pin down the $\mathcal{O}_K$-module structure of $H^*_{\mathrm{dR}}(BG/\mathcal{O}_K)$, we use the fact that $H^i_{\mathrm{dR}}(BG/\mathcal{O}_K)/\pi$ injects into $H^i_{\mathrm{dR}}(B\alpha_p/\mathbb{F}_p)$ which is always one-dimensional for $i \geq 0$ due to [ABM19, Proposition 4.10].

Lastly, pick a preimage of β under $H^2_{\mathrm{dR}}(BG/\mathcal{O}_K) \twoheadrightarrow H^2(BG, \mathcal{O})$, denote it by $\beta' \in H^2_{\mathrm{dR}}(BG/\mathcal{O}_K)$. Since $H^*_{\mathrm{dR}}(BG/\mathcal{O}_K) \to H^*_{\mathrm{dR}}(B\alpha_p/\mathbb{F}_p)$ is a map preserving multiplication, we see that β'^n is a generator of $H^{2n}_{\mathrm{dR}}(BG/\mathcal{O}_K)$. This finishes the proof of the ring structure on $H^*_{\mathrm{dR}}(BG/\mathcal{O}_K)$. ☐

Similarly, we can understand the Hodge–Tate spectral sequence of BG with the aid of [ABM19, Remark 4.13].

Proposition 4.8. *In the Hodge–Tate spectral sequence of BG, we have (up to unit) $d_2(\tau) = \beta^2$ and $d_2(\beta) = d_2(u) = 0$ for all p. The Hodge–Tate cohomology of BG is given by*

$$H^*_{\mathrm{HT}}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K[u']/(pu'),$$

where u' has degree 2.

Proof. The proof is almost the same as the proof of Proposition 4.7, except we now have $d_2(\alpha) = d_2(\beta) = d_2(u) = 0$ and $d_2(s) = \beta$ in the special fiber. ☐

In particular, both of the Hodge–de Rham and Hodge–Tate spectral sequences are non-degenerate with nonzero differentials starting at degree 3, and the Hodge (resp. conjugate) filtrations on de Rham (resp. Hodge–Tate) cohomology is not split starting at degree 2. When the prime is $p \geq 11$, these give rise to stacky examples satisfying condition (2) of the main theorem which violates the conclusion. The obstruction of lifting G to $\mathrm{Spec}(\mathfrak{S}/(E^2))$ specializes (under modulo u) to the obstruction of lifting α_p to $\mathrm{Spec}(W_2)$, which is nonzero.

We can also determine the prismatic cohomology of BG using Proposition 4.8. Before that, we need a few words about prismatic cohomology of a smooth proper stack. Note that Bhatt–Scholze showed that the prismatic cohomology satisfies (quasi-)syntomic descent [BS19, Theorem 1.15.(2)], therefore one can define the prismatic cohomology of BG as that of the sheaf $\mathbb{\Delta}_{-/\mathfrak{S}}$ on the (quasi-)syntomic site of BG, c.f. [ABM19, Section 2]. All the results stated previously (e.g. Theorem 3.7 and Theorem 3.8) still hold verbatim.

Proposition 4.9. *The prismatic cohomology of BG is given by*

$$H^*_{\mathbb{\Delta}}(BG/\mathfrak{S}) \simeq \mathfrak{S}[\tilde{u}]/(p\tilde{u}),$$

where $\tilde{u}$ has degree 2.

Before giving the proof, we need an auxiliary lemma.

Lemma 4.10. *Let M be a cyclic torsion Breuil–Kisin module over $\mathfrak{S}$ with no E-torsion, then there is an integer n such that $M \simeq \mathfrak{S}/(p^n)$.*

Proof. Say $M = \mathfrak{S}/I$. First we know that $M[1/p] = 0$, see [BMS18, Proposition 4.3]. Since M has no E-torsion, we see that M contains no nonzero finite $\mathfrak{S}$-submodule. Let n be the smallest integer such that $p^n \in I$. It suffices to show that for any non-unit $f \in \mathfrak{S} \setminus (p)$, the smallest integer m such that $p^m f \in I$ is n. Suppose otherwise, then we have $m < n$. Then the image of

$$\mathfrak{S}/(f,p) \xrightarrow{\cdot p^{n-1}} M$$

is a nonzero (as $p^{n-1} \notin I$) finite (as the image of f in $\mathfrak{S}/(p) = k[\![u]\!]$ is nonzero and non-unit) submodule, which we have argued is impossible. Therefore we must have $n = m$. ☐

Proof of Proposition 4.9. The Hodge–Tate specialization gives us short exact sequences:

$$0 \to H^*_{\Delta}(BG/\mathfrak{S})/(E) \to H^*_{\mathrm{HT}}(BG/\mathcal{O}_K) \to H^{*+1}_{\Delta}(BG/\mathfrak{S})[E] \to 0,$$

where $M[E]$ denotes the E-torsion of an $\mathfrak{S}$-module M. By our Proposition 4.8 and Nakayama's Lemma, we see that

(1) the odd degree prismatic cohomology groups of BG are zero; and
(2) the positive even degree prismatic cohomology groups of BG are cyclic and E-torsion free.

Applying the above lemma, we see that $H^2_{\Delta}(BG/\mathfrak{S}) \simeq \mathfrak{S}/(p^n)$ for some n. To see that n must be 1, we use the fact that it is so under the Hodge–Tate specialization. Powers of any generator of $H^2_{\Delta}(BG/\mathfrak{S})$ must still be generators of the corresponding prismatic cohomology group, as it is so after the Hodge–Tate specialization.

☐

Remark 4.11. We do not know how Frobenius acts on the prismatic cohomology groups. Since the geometric generic fiber of BG is $B\mathbb{Z}/p$, we at least know that the Frobenius is not identically zero, by étale specialization of the prismatic cohomology [BS19, Theorem 1.8.(4)]. On the other hand, since Frobenius is zero for $B\alpha_p$, we know that Frobenius also cannot be surjective on $H^2_{\Delta}(BG/\mathfrak{S})$. Since $((\mathbb{F}_p[\![u]\!])^*)^{p-1} = (1 + (u), \times)$, we see that after choosing an appropriate generator, the Frobenius on $H^2_{\Delta}(BG/\mathfrak{S}) \simeq \mathfrak{S}/(p) \cong \mathbb{F}_p[\![u]\!]$ sends 1 to $\gamma \cdot u^d$, where $\gamma \in \mathbb{F}_p^*$ and d is a positive integer. It would be interesting to understand the relation between our choice[3] of π and the values γ and d.

4.3. **Approximating BG.** In this last subsection, let us show that the pathologies of BG is inherited by approximations of BG, so that in the end we can get some scheme examples. For this purpose, it suffices to follow [ABM19, Section 6].

Proposition 4.12 (See also [ABM19, Theorem 1.2])**.** *For any integer $d \geq 0$, there exists a smooth projective $\mathcal{O}_K$-scheme $\mathcal{X}$ of dimension d together with a map $\mathcal{X} \to BG$ such that the pullback $H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \to H^i(\mathcal{X}, \wedge^j L_{\mathcal{X}/\mathcal{O}_K})$ is injective for $i + j \leq d$.*

Proof. We simply follow the first paragraph of [ABM19, Section 6, proof of Theorem 1.2]. By standard argument (see e.g. [BMS18, 2.7-2.9]), we can find an integral representation V of G and a d-dimensional complete intersection $\mathcal{Y} \subset \mathbb{P}(V)$ such that $\mathcal{Y}$ is stable under the G-action, the action is free, and $\mathcal{X} := \mathcal{Y}/G \simeq [\mathcal{Y}/G]$ is

[3] Recall that we need to make such a choice in order to lift α_p.

smooth and projective over $\mathcal{O}_K$ together with a map $\mathcal{X} \to BG$. We see that the special fiber of this map induces injections on the corresponding Hodge cohomology groups. Now we observe that the composite map

$$H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \to H^i(B\alpha_p, \wedge^j L_{B\alpha_p/\mathbb{F}_p}) \to H^i(\mathcal{X}_0, \Omega^j_{\mathcal{X}_0/\mathbb{F}_p}))$$

is injective when $i + j \leq d$ (as it is composite of two injective maps) and factors through $H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \to H^i(\mathcal{X}, \Omega^j_{\mathcal{X}/\mathcal{O}_K})$. Hence the latter map must also be injective when $i + j \leq d$. □

By choosing $d = 4$ and using Proposition 4.7 and Proposition 4.8, we arrive at the following theorem.

Theorem 4.13. *There exists a smooth projective relative 4-fold $\mathcal{X}$ over a ramified degree two extension $\mathcal{O}_K$ of $\mathbb{Z}_p$, such that both of its Hodge–de Rham and Hodge–Tate spectral sequences are non-degenerate. Moreover the Hodge/conjugate filtrations are non-split as $\mathcal{O}_K$-modules.*

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