AN EXAMPLE OF LIFTINGS WITH DIFFERENT HODGE NUMBERS

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ABSTRACT. In this paper, we exhibit an example of a smooth proper variety in positive characteristic possessing two liftings with different Hodge numbers.

1. INTRODUCTION

Does a smooth proper variety in positive characteristic know the Hodge number of its liftings? In this paper, we construct an example showing that the answer is no in general. There are some constraints to make such an example. Such an example must be of dimension at least 3 (see Proposition 3.8). The examples we constructed here are 3-folds in all characteristics (including characteristic 2), see Section 2, Subsection 3.1 and Subsection 3.2.

2. EXAMPLES FOR $p \geq 5$

In this section, let $p \geq 5$ be a prime, let $\mathrm{R} = \mathbb{Z}_p[\zeta_p]$ where ζ_p is a primitive p-th root of unity. Let $\mathcal{E}/\mathrm{Spec}(\mathrm{R})$ be an ordinary elliptic curve possessing a p-torsion $P \in \mathcal{E}(\mathrm{R})[p]$ which does not specialize to identity element. There are such pairs over $\mathbb{Z}_p$. Indeed, the Honda–Tate theory tells us the polynomial $x^2 - x + p$ corresponds to an ordinary elliptic curve $\mathcal{E}_0$ over $\mathbb{F}_p$ with p rational points (c.f. [Tat71, THÉORÉME 1.(i)]). In particular, we see that $\mathcal{E}_0(\mathbb{F}_p) \cong \mathbb{Z}/p$. Now the Serre–Tate theory (c.f. [Kat81, Chapter 2]) tells us $\mathcal{E}$, the canonical lift of $\mathcal{E}_0$ over $\mathbb{Z}_p$, satisfies $\mathcal{E}[p]^{\acute{e}t} \cong \mathbb{Z}/p \times \mathrm{Spec}(\mathbb{Z}_p)$. Hence we see that all the rational points of $\mathcal{E}_0$ are liftable over $\mathbb{Z}_p$. Fix such an auxiliary elliptic curve along with this p-torsion point. Denote the uniformizer $\zeta_p - 1 \in \mathrm{R}$ by π. Denote the fraction field of R by K, the residue field by κ.

We use curly letters to denote integral objects over Spec(R), use the corresponding straight letter to denote its generic fibre and use subscript $(\cdot)_0$ to denote its special fibre, i.e., reduction mod π. For example, we will denote the generic fibre of $\mathcal{E}$ by E and the special fibre by $\mathcal{E}_0$. To simplify the notations, whenever no confusion seems to arise, we will not denote the base over which we make the fibre product.

Let $\mathcal{C}$ be the proper smooth hyper-elliptic curve over Spec(R) defined by

$$v^2 = \sum_{i=0}^{p-1} \frac{\binom{p}{i}}{(\zeta_p - 1)^i} u^{p-i}.$$

We leave it to the readers to verify that this indeed defines a smooth proper curve with the other affine piece given by $v^2 = \sum_{i=0}^{p-1} \frac{\binom{p}{i}}{(\zeta_p-1)^i} u^{i+1}$.

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One checks easily that this curve has genus $\frac{p-1}{2}$ and $\mathcal{C}_0$, its reduction mod π, is the hyper-elliptic curve defined by

$$v^2 = u^p - u.$$

After inverting π and making the substitution

$$x = (\zeta_p - 1)u + 1$$
$$y = v,$$

we see that C, the generic fibre of $\mathcal{C}$, is the hyper-elliptic curve defined by

$$(\zeta_p - 1)^p y^2 = x^p - 1.$$

There is an R-linear $\mathbb{Z}/p = \langle\sigma\rangle$-action on $\mathcal{C}$ given by

$$\sigma(u) = \zeta_p \cdot u + 1$$
$$\sigma(v) = v.$$

One checks that in the generic fibre, using xy-coordinate, this action becomes $\sigma(x) = \zeta_p \cdot x$ and $\sigma(y) = y$. In the special fibre, this action becomes $\sigma(u) = u + 1$ and $\sigma(v) = v$.

We have a canonical character $\mathbb{Z}/p \to K^\times$ given by

$$\chi\colon \langle\sigma\rangle \to K^\times$$
$$\sigma \mapsto \zeta_p.$$

Proposition 2.1. *Using notations as above, we have*

(1) *in the special fibre, the action of σ and σ^4 are conjugate by an automorphism of $\mathcal{C}_0$;*
(2) *in the generic fibre, we have a decomposition*

$$H^0(C, \Omega^1) = \bigoplus_{1 \le i \le \frac{p-1}{2}} \chi^i$$

as representations of $\langle\sigma\rangle$.

Proof. (1) Consider the automorphism $\tau\colon \mathcal{C}_0 \to \mathcal{C}_0$ given by

$$\tau(u) = 4u$$
$$\tau(v) = 2v.$$

One easily verifies that this preserves the equation $v^2 = u^p - u$ hence an automorphism of $\mathcal{C}_0$, and that $\tau \circ \sigma \circ \tau^{-1} = \sigma^4$. This completes the proof of (1).

(2) Recall that $\left\{\frac{dx}{y}, \frac{x\,dx}{y}, \ldots, \frac{x^{g-1}\,dx}{y}\right\}$ form a basis of $H^0(C, \Omega^1)$ whenever C is a genus g hyper-elliptic curve given by $y^2 = f(x)$ [GH94, page 255]. One checks immediately that under this basis, σ acts by the characters as in the Proposition. □

Recall that we have fixed an auxiliary elliptic curve $\mathcal{E}$ over R and a p-torsion point P on it which does not specialize to identity element. Hence translating by P defines an order p automorphism of $\mathcal{E}$ over R which acts trivially on the global 1-forms, let us denote this action by τ_P.

Construction 2.2. Let $\mathcal{X} := (\mathcal{C} \times \mathcal{C} \times \mathcal{E})/\langle(\sigma, \sigma, \tau_P)\rangle$ and let $\mathcal{Y} := (\mathcal{C} \times \mathcal{C} \times \mathcal{E})/\langle(\sigma, \sigma^4, \tau_P)\rangle$. Here we mean the schematic quotient by the indicated *diagonal* action.

Then we have the following:

Proposition 2.3. *Both* $\mathcal{X}$ *and* $\mathcal{Y}$ *are smooth projective over* Spec(R)*, and their special fibers are isomorphic as smooth proper* k*-varieties. Moreover we have* $H^0(X, \Omega^3_X) = 0$ *and* $H^0(Y, \Omega^3_Y) \neq 0$*.*

Proof. The third component ensures that the action is fixed point free. Therefore the quotient is smooth and proper, and it satisfies the following base change of taking quotient:

$$\mathcal{X}_0 \cong (\mathcal{C}_0 \times \mathcal{C}_0 \times \mathcal{E}_0)/\langle(\sigma, \sigma, \tau_P)\rangle$$
$$\mathcal{Y}_0 \cong (\mathcal{C}_0 \times \mathcal{C}_0 \times \mathcal{E}_0)/\langle(\sigma, \sigma^4, \tau_P)\rangle.$$

By 2.1 (1), σ and σ^4 are conjugate to each other by τ (with notations loc. cit.). We see that (id, τ, id) induces an isomorphism between $\mathcal{X}_0$ and $\mathcal{Y}_0$.

In the generic fibre, we have that the global 3-forms of the quotient are identified as the invariant (regarding respective actions) global 3-forms of $C \times C \times E$. By Künneth formula and 2.1 (2), we have the following decomposition

$$H^{3,0}(C \times C \times E) = \bigoplus_{1 \leq i \leq \frac{p-1}{2}} \chi^i \otimes \bigoplus_{1 \leq i \leq \frac{p-1}{2}} \chi^i \otimes \mathbb{1}$$

as (σ, σ, τ_P)-representations. Therefore we see that $H^{3,0}(X) = 0$. To see that $H^0(Y, \Omega^3_Y) \neq 0$, we note that in the above decomposition $\frac{x_1\, dx_1}{y_1} \wedge \frac{x_2^{\frac{p-3}{2}}\, dx_2}{y_2} \wedge \omega$ is invariant under $(\sigma, \sigma^4, \tau_P)$, where ω is some translation invariant nonzero 1-form on E. Here we have used $p \geq 5$, so that $\frac{x_1\, dx_1}{y_1}$ is a *holomorphic* global 1-form on C. Hence we get that $H^0(Y, \Omega^3_Y) \neq 0$. $\square$

Remark 2.4. One may compute the hodge diamonds of X and Y, let us record the result here. The Hodge diamond of X is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 0 & & p+2 & & 0 & \\ 0 & & p+1 & & p+1 & & 0 \\ & 0 & & p+2 & & 0 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array}$$

and the Hodge diamond of Y is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & \frac{p-1}{4}+\epsilon_p & & \frac{p+5}{2}-2\epsilon_p & & \frac{p-1}{4}+\epsilon_p & \\ \frac{p-1}{4}+\epsilon_p & & \frac{3p+5}{4}-\epsilon_p & & \frac{3p+5}{4}-\epsilon_p & & \frac{p-1}{4}+\epsilon_p \\ & \frac{p-1}{4}+\epsilon_p & & \frac{p+5}{2}-2\epsilon_p & & \frac{p-1}{4}+\epsilon_p & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array}$$

where ϵ_p depends on the congruent class of p mod 8, and is given by

$$\epsilon_p = \begin{cases} 0 & p \equiv 1 \mod 8 \\ -\frac{1}{2} & p \equiv 3 \mod 8 \\ 1 & p \equiv 5 \mod 8 \\ \frac{1}{2} & p \equiv 7 \mod 8 \end{cases}.$$

Remark 2.5. Those readers who are familiar with Deligne–Lusztig varieties perhaps have realized the curve $\mathcal{C}_0 = \{y^2 = x^p - x\}$ is nothing but the quotient of the Drinfel'd curve $\{y^{p+1} = x^p z - xz^p\}$ (c.f. [DL76, Ch. 2]), where the quotient is with respect to the subgroup $\mu_{\frac{p+1}{2}} \subset \mu_{p+1}$ acts on y by multiplication and fixes x and z. Hence the curve $\mathcal{C}_0$ bears the action of $\mathrm{SL}_2(\mathbb{F}_p) \times \mathbb{Z}/2$ where the second factor is the hyper-elliptic structure of $\mathcal{C}_0$. Under this identification, the σ (resp. τ) we find above correspond to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ (possibly times the nontrivial involution depending on whether $2^{\frac{p+1}{2}} = 2$ or -2 in $\mathbb{F}_p$)).

Following the same spirit, we construct similar example in the case $p = 3$ (see Subsection 3.1) and $p = 2$ (see Subsection 3.2).

3. Complements and Remarks

3.1. **Case $p = 3$.** Let us consider the case $p = 3$ in this subsection. Let $\mathrm{R} = \mathbb{Z}_3[\omega, i]$ where ω is a 3-rd root of unity and $i^2 = -1$. Denote the uniformizer $\omega - 1 \in \mathrm{R}$ by π. Let $\mathcal{C}$ be the proper smooth hyper-elliptic curve over $\mathrm{Spec}(\mathrm{R})$ defined by

$$v^2 = (u^3 + (\omega^2 - 1)u^2 - \omega^2 u)^3 + (u^3 + (\omega^2 - 1)u^2 - \omega^2 u).$$

One checks easily that this curve has genus 4 and $\mathcal{C}_0$, its reduction mod π, is the hyper-elliptic curve defined by

$$v^2 = u^9 - u.$$

After inverting π and making the substitution

$$x = (\omega - 1)u + 1$$
$$y = v,$$

we see that C, the generic fibre of $\mathcal{C}$, is the hyper-elliptic curve defined by

$$y^2 = \frac{1}{(\omega-1)^9} \cdot (x^3-1)^3 + \frac{1}{(\omega-1)^3} \cdot (x^3-1).$$

There is an R-linear $\mathbb{Z}/3$-action on $\mathcal{C}$ given by

$$\sigma(u) = \omega \cdot u + 1$$
$$\sigma(v) = v.$$

Similar to the Section 2 and use analogous notation as there, we state the following:

Proposition 3.1. *Using notations as above, we have*

(1) *in the special fibre, the action of σ and σ^2 are conjugate by an automorphism of $\mathcal{C}_0$;*
(2) *in the generic fibre, we have a decomposition*

$$H^0(C, \Omega^1_C) = \chi^{\oplus 2} \oplus \chi^2 \oplus \mathbb{1}$$

as representations of $\langle\sigma\rangle$.

The proof is similar, notice that now the automorphism group of $\mathcal{C}_0$ is $\mathrm{SL}_2(\mathbb{F}_9) \times \mathbb{Z}/2$ and $2 = -1 = i^2$ is a square in $\mathbb{F}_9$.

Possibly passing to an unramified extension of R, we may assume as before that there is an elliptic curve $\mathcal{E}$ over R together with a nonzero 3-torsion point P. Then we make the following:

Construction 3.2. Let $\mathcal{X} := (\mathcal{C} \times \mathcal{C} \times \mathcal{E})/\langle(\sigma, \sigma, \tau_P)\rangle$ and let $\mathcal{Y} := (\mathcal{C} \times \mathcal{C} \times \mathcal{E})/\langle(\sigma, \sigma^2, \tau_P)\rangle$.

Proposition 3.3. *Both of $\mathcal{X}$ and $\mathcal{Y}$ are smooth projective over* $\mathrm{Spec}(\mathrm{R})$ *and we have $h^{3,0}(X) = 5$ and $h^{3,0}(Y) = 6$.*

3.2. **Case $p = 2$.** Let us consider the case $p = 2$ in this subsection. Let us just construct such an example over some 2-adic base (without caring how ramified this base is). Let $\mathcal{O}$ be the ring of integers inside a large enough local 2-adic field K so that there are

(1) an elliptic curve with ordinary reduction $\mathcal{E}$ over $\mathrm{Spec}(\mathcal{O})$ and a 4-torsion point $P \in \mathcal{E}(\mathcal{O})[4]$ such that $2 \cdot P_0 \neq 0$, and;
(2) an elliptic curve $\mathcal{C}$ over $\mathrm{Spec}(\mathcal{O})$ with j-invariant 1728 such that there is an automorphism of $\mathcal{C}$ of order 4 which will be denoted by i and so that $\mathrm{Aut}(\mathcal{C}_0) = \mathcal{O}_D^*$. Here D denotes the quaternion algebra over $\mathbb{Q}$ ramified over 2 and ∞, and $\mathcal{O}_D^*$ means the group of units inside the maximal order of this quaternion algebra.

One can always enlarge the 2-adic field K so that these are achieved. Note that by the last condition, the primitive fourth root of unity must lie in K and let us still denote it by i. Finally there is a tautological character $\chi : \mathbb{Z}/4 \to K^*$ sending 1 to i. The following Proposition is what we need.

Proposition 3.4. *Using notations as above, we have*

(1) *in the special fibre, the action of i and $-i$ are conjugate by an automorphism of $\mathcal{C}_0$;*

(2) *in the generic fibre, we have*

$$H^0(C, \Omega^1_C) = \chi$$

as representations of $\mathbb{Z}/4 \cong \langle i \rangle$.

This is almost trivial: for (1) we have the identity $-j \cdot i \cdot j = -i$, and (2) is a standard fact about elliptic curve with complex multiplication by i.

Lastly we make the following:

Construction 3.5. Let $\mathcal{X} := (\mathcal{C} \times \mathcal{C} \times \mathcal{E})/\langle (i, i, \tau_P) \rangle$ and let $\mathcal{Y} := (\mathcal{C} \times \mathcal{C} \times \mathcal{E})/\langle (i, -i, \tau_P) \rangle$.

Proposition 3.6. *Both of* $\mathcal{X}$ *and* $\mathcal{Y}$ *are smooth projective over* $\mathrm{Spec}(\mathcal{O})$ *and we have* $h^{3,0}(X) = 0$ *and* $h^{3,0}(Y) = 1$.

Remark 3.7. Note that in characteristic 3, the automorphism group of the elliptic curve with j-invariant $0 = 1728$ is the dicyclic group Dic_3 of order 12. In particular, the automorphism ω is conjugate to ω^2. Using this, we may make similar examples in characteristic 3.

3.3. **Final Remarks.** The following Proposition shows that our example is sharp in terms of its dimension (the case of curve is trivial).

Proposition 3.8. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be smooth proper schemes over* $\mathrm{Spec}(\mathcal{O})$ *of relative dimension* 2. *Suppose* $\mathcal{X}_0 \cong \mathcal{Y}_0$, *then* $h^{i,j}(X) = h^{i,j}(Y)$ *for all* i,j.

Proof. Since for surfaces we have $\frac{1}{2}b_1 = h^{0,1} = h^{1,0} = h^{0,3} = h^{3,0}$, by smooth proper base change we know that these numbers only depend on the special fibre. Therefore the Hodge numbers of X and Y agree except for the degree 2 part. Now the fact that the Euler characteristic of a flat coherent sheaf stays constant in a family shows that the degree 2 Hodge numbers of X and Y also agree. □

In order to make such an example, dimension is certainly not the only constraint.

Proposition 3.9. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be smooth proper schemes over* $\mathrm{Spec}(\mathcal{O})$ *with* $\mathcal{X}_0 \cong \mathcal{Y}_0$. *Suppose the Hodge-to-de Rham spectral sequence for* $\mathcal{X}_0$ *degenerates at* E_1*-page and* $H^r_{crys}(\mathcal{X}_0/W(k))$ *is torsion-free for all* r. *Then* $h^{i,j}(X) = h^{i,j}(Y)$ *for all* i,j.

Proof. The crystalline cohomology being torsion-free implies that $h^r_{\mathrm{dR}}(X) = h^r_{\mathrm{dR}}(\mathcal{X}_0)$. In the generic fibre, by Hodge theory, we have $\sum_{i+j=r} h^{i,j}(X) = h^r_{\mathrm{dR}}(X)$. In the special fibre, by the degeneration of Hodge-to-de Rham spectral sequence, we have $\sum_{i+j=r} h^{i,j}(\mathcal{X}_0) = h^r_{\mathrm{dR}}(\mathcal{X}_0)$. These three equalities along with upper semi-continuity of $h^{i,j}$ imply $h^{i,j}(\mathcal{X}_0) = h^{i,j}(X)$. Then same argument implies $h^{i,j}(\mathcal{X}_0) = h^{i,j}(Y)$. Hence we see that the Hodge numbers of X and Y are the same. □

Remark 3.10. Using the fact that $H_1(C;\mathbb{Z})$ as a $\mathbb{Z}/p$-module is the augmentation ideal in $\mathbb{Z}[\mathbb{Z}/p]$, one can show that $h^1_{\mathrm{dR}}(\mathcal{X}_0) = 4$ and $h^1_{\mathrm{dR}}(X) = 2$, which implies that $\dim_{\mathbb{F}_p} H^2_{\mathrm{crys}}(\mathcal{X}_0/W(k))[p] = 2$.

A more detailed study shows that the length of torsions in the crystalline cohomology groups of our examples stay bounded for all primes p, however the discrepancy between $h^{3,0}(X)$ and $h^{3,0}(Y)$ grows linearly in p.

Remark 3.11. Although our examples here are not simply connected, one can bootstrap them to simply connected ones by embedding them into a projective space, blow up, and take complete intersections of dimension at least 3. The author would like to thank Jason Starr for pointing this out to him.

We conclude this paper by observing that the examples we found are over ramified base with absolute ramification index $p-1$ and asking:

Question 3.12. Is there a pair of smooth proper schemes $\mathcal{X}$ and $\mathcal{Y}$ over $\mathrm{Spec}(W(k))$, such that

(1) $\mathcal{X}_0 \cong \mathcal{Y}_0$ and;
(2) $h^{i,j}(X) \neq h^{i,j}(Y)$ for some i, j?

Note that by [DI87, Corollaire 2.4] the Hodge-to-de Rham spectral sequence for any smooth proper $\mathcal{X}_0$ degenerates at E_1-page, provided that $\dim(\mathcal{X}_0) < p$ and $\mathcal{X}_0$ lifts to $W_2(k)$. In particular, the example asked for in Question 3.12, if it exists and is of small dimension (say, 3-fold), must have torsion in $H^*_{\mathrm{crys}}(\mathcal{X}_0/W(k))$ by Proposition 3.9.

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