LOGARITHMIC DE RHAM COMPARISON FOR OPEN RIGID SPACES

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Abstract. In this note, we prove the logarithmic p-adic comparison theorem for open rigid analytic varieties. We prove that a smooth rigid analytic variety with a strict simple normal crossing divisor is locally $K(\pi, 1)$ (in a certain sense) with respect to $\mathbb{F}_p$-local systems and ramified coverings along the divisor. We follow Scholze's method to produce a pro-version of the Faltings site and use this site to prove a primitive comparison theorem in our setting. After introducing period sheaves in our setting, we prove aforesaid comparison theorem.

Contents

1. Introduction

Historically, classical Hodge theory was developed from Hodge's results up through Deligne's papers on mixed Hodge structures in the early 1970's. The famous decomposition theorem is the following.

Theorem 1.1 (Hodge, Deligne). *Let X be a smooth proper variety over complex numbers $\mathbb{C}$ with a strict simple normal crossing divisor D. Then we have*

$$H^m_{sing}(X - D, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^m(X, \Omega^\bullet_X(\log D)) \cong \bigoplus_{i+j=m} H^i(X, \Omega^j_X(\log D)),$$

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where $\Omega^j_X(\log D)$ *is the sheaf of* j*-forms with logarithmic singularities along* D *on* X*.*

The p-adic Hodge theory properly began around 1966 when Tate [Tat67, p. 180 Remark] proved a p-adic version of the comparison theorem for an abelian variety of good reduction over a p-adic field. After the works of Fontaine, Messing, Bloch, Kato, etc., Faltings proved the following.

Theorem 1.2. [Fal88] *Let* X *be a smooth proper variety over a* p*-adic field* k *with a strict simple normal crossing divisor* D*. Then there exists a* $\mathrm{Gal}(\overline{k}/k)$*-equivariant isomorphism*

$$H^m_{\text{ét}}((X-D)_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=m} H^i(X, \Omega^j_X(\log D)) \otimes_K \mathbb{C}_p(-j),$$

where $\Omega^j_X(\log D)$ *is the sheaf of* j*-forms with logarithmic singularities along* D *on* X *and* $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$*.*

Afterwards, many people have found other ways to produce this comparison isomorphism. There is another remarkable approach to prove such a comparison, due to Beilinson (see [Bei12]), using derived de Rham cohomology (of Illusie), h-topology and de Jong's alterations.

Recently, Scholze proved a stronger version of Theorem 1.2, namely, the de Rham comparison theorem for a smooth proper rigid analytic space over a p-adic field. Moreover, the comparison theorem that he proved allows coefficients to be local systems, see [Sch13a, Theorem 8.4]. However, his theorem does not include the logarithmic case. The purpose of this note is to prove the de Rham comparison in the logarithmic case (for constant coefficients) using the same methods.

It is also worth mentioning that in the work of Colmez–Nizioł, they proved a semistable comparison for semistable formal log-schemes (see [CN17, Corollary 5.26]). In particular, they've already obtained the de Rham comparison in the logarithmic case (for constant coefficients) assuming the appearance of a semistable formal model.

Let k be a discretely valued complete non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field κ. Our main comparison theorem (see Theorem 7.9 and Theorem 7.14) is the following:

Theorem 1.3. *Let* X *be a proper smooth adic space over* $\mathrm{Spa}(k, \mathcal{O}_k)$ *with a strict simple normal crossing divisor* D *and complement* $U := X \setminus D$*. Then, there is a natural* $\mathrm{Gal}(\overline{k}/k)$*-equivariant isomorphism*

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i(X, \Omega^\bullet_X(\log D)) \otimes_k B_{\mathrm{dR}}$$

preserving filtrations. Moreover, the logarithmic Hodge–de Rham spectral sequence

$$E_1^{j,i} = H^i(X, \Omega^j_X(\log D)) \Longrightarrow H^{i+j}(X, \Omega^\bullet_X(\log D))$$

degenerates. In particular, the logarithmic Hodge–Tate spectral sequence also degenerates and yields the logarithmic Hodge–Tate decomposition

$$H^m(U_{\overline{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \hat{\overline{k}} \cong \bigoplus_j H^{m-j}(X, \Omega^j_X(\log D)) \otimes_k \hat{\overline{k}}(-j).$$

During the preparation of this note, we learned that Hansheng Diao, Kai-Wen Lan, Ruochuan Liu and Xinwen Zhu have proved a more powerful version of this comparison theorem including allowing more general coefficients (see [DLLZ18, Theorem 1.1]).

We hope our approach following Scholze and using the Faltings site is still interesting in its own.

In the rest of this introduction, we give a brief descriptions of the organization of this note. In Subsection 2.3 we introduce the Faltings site $X_{\log}$ and show that the complement of a strict simple normal crossing divisor is locally $K(\pi, 1)$ (in a certain sense) with respect to $\mathbb{F}_p$-local systems see Theorem 2.8 and Proposition 2.12.[1] The main consequence is that we can compute the cohomology of local systems on $U_{\text{ét}}$ via $X_{\log}$. The main ingredients for the proofs are results of Lütkebohmert in [Lüt93], Scholze's $K(\pi, 1)$-result for affinoid spaces and Gysin sequence.

In Section 3, we introduce a general method to produce a pro-site X_{prolog} of the Faltings site $X_{\log}$. This general method is recapturing [Sch13a, Section 3]. We also show that the pro-site X_{prolog} shares a lot of good properties, e.g. algebraicity and it has a coherent terminal object if the rigid space X is proper over k. Most of the arguments are formal and similar to counterparts in [Sch13a, Section 3].

In Section 4, we introduce structure sheaves on $X_{\log}$ and X_{prolog}. We also show that X_{prolog} has affinoid perfectoid basis, see Lemma 4.8. The main difference of X_{prolog} from the pro-étale site is that we are allowed to take any root (not just p-root) of the coordinates defining the divisor D, see Example 4.4. This difference is clear from [Fal88].

In Section 5, we follow the method of Scholze to show the primitive comparison Theorem 5.1 in our setting. A similar result has been obtained by Diao in the setting of (pro)-Kummer étale site, see [Dia17, Proposition 4.4]. To show the comparison theorem for X_{prolog}, we need to enhance some Scholze's results in the case allowing ramified coverings.

In Section 6, we introduce the period sheaves on X_{prolog}. The new ingredient is a logarithmic version of the period sheaf, $\mathcal{O}\mathbb{B}^+_{\text{logdR}}$. The main result of this section is the logarithmic Poincaré Lemma, see Corollary 6.10.

In Subsection 7.1, we introduce a notion of vector bundles on the Faltings site $X_{\log}$ and prove Theorem A and Theorem B à la Cartan for them. Then in Subsection 7.2 we prove the aforesaid de Rham comparison theorem.

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[1]Shortly after posting the first version of this note on arXiv, we are pointed out that similar result has been obtained by Colmez–Nizioł (see [CN17, 5.1.4]) via similar methods.

We thank Hansheng Diao, Kai-Wen Lan, Ruochuan Liu and Xinwen Zhu for communicating with us. We especially thank Xinwen Zhu heartily for many precious comments and advice on an early draft of this note.

We also thank Pierre Colmez and Wiesława Nizioł for communications concerning the first version of this note.

Notations and Conventions. In this note, unless specified otherwise, we will use the following notations and conventions. Let k be a p-adic field, i.e., discretely valued complete non-archimedean field extension of $\mathbb{Q}_p$ with perfect residue field. We denote its ring of integers by $\mathcal{O}_k$. We will use K to denote a perfectoid field which is the completion of some algebraic extension of k.

Let X be a smooth proper rigid space over k of dimension n and let $D = \bigcup_{i\in I} D_i \subset X$ be a divisor, here I is a finite index set. For any subset $J \subset I$, we use D^J to denote $\bigcap_{i\in J} D_i$. We say D is a strict simple normal crossing (shorthand by SSNC from now on) divisor if all of D^J's are smooth of codimension $|J|$ where $|J| :=$ number of elements in J. Here D^J has codimension greater than n means it is empty. We denote $X \setminus D$ by U. For any rigid space $V \to X$ admitting a map to X, we denote the preimage of U by V°.

We use notation $\mathbb{D}^r(\underline{T})$ to denote r-dimensional unit polydisc with coordinates given by T_i. We denote $\mathbb{D}^r(\underline{T}) \setminus V(T_1 \cdots T^r)$ by $\mathbb{D}^{\circ,r}(\underline{T})$.

Let A be a ring, we denote its normalization by A^ν. If $f_1, \ldots, f_r$ are r elements in an affinoid algebra A, then we denote $\mathrm{Sp}((A[\sqrt[m]{f_1}, \ldots, \sqrt[m]{f_r}])^\nu)$ by $\mathrm{Sp}(\mathrm{A}[\sqrt[m]{f_l}])$.

We use both the language of adic spaces of finite type over $\mathrm{Spa}(k, \mathcal{O}_k)$ and rigid spaces over $\mathrm{Sp}(k)$ interchangeably, we hope this does not confuse the reader.

2. Preliminaries

2.1. **Abhyankar's Lemma.** Let us discuss Abhyankar's Lemma for rigid spaces over p-adic fields. This is more or less already obtained by Lütkebohmert in [Lüt93], see also [Han17, Section 2.2].

Proposition 2.1. *Let S be a smooth rigid space over k which is not necessarily quasi-compact or quasi-separated. Let*

$$\phi\colon Y \to X = S \times \mathbb{D}^{\circ,r}(\underline{z})$$

be a finite étale covering of degree d. Then after pulling Y back to $S \times \mathbb{D}^{\circ,r}(\underline{T})$ along

$$S \times \mathbb{D}^{\circ,r}(\underline{T}) \to S \times \mathbb{D}^{\circ,r}(\underline{z}),\ z_i \mapsto T_i^{d!}$$

it extends to a finite étale covering of $S \times \mathbb{D}^r(\underline{T})$.

Proof. Step 1: let us first prove this in the case where $r = 1$. Since extension of covering is faithful (see [Han17, Proposition 2.9]), by descent it suffices to prove the statement after replacing S by an étale cover of S. Therefore we may assume that the conditions of [Lüt93, Lemma 3.2] are satisfied. Our statement just follows from [Lüt93, Lemma 3.2].

Step 2: let us prove the general case by induction on r. Write

$$S \times \mathbb{D}^{\circ,r} = S \times (\mathbb{D}^{\circ,r-1}) \times (\mathbb{D}\backslash\{0\}).$$

By step 1 we see that after pulling Y back along

$$S \times (\mathbb{D}^{\circ,r-1}(\underline{z})) \times (\mathbb{D}(T_n)\backslash\{0\}) \to S \times (\mathbb{D}^{\circ,r-1}(\underline{z})) \times (\mathbb{D}(z_n)\backslash\{0\}),\ z_n \mapsto T_n^{d!}$$

it extends to a finite étale covering of $S \times (\mathbb{D}^{\circ, r-1}(\underline{z})) \times \mathbb{D}(T_n)$. Now by induction, we are done. □

From the proposition above, we can deduce the following Theorem which can be thought of as the analogue of Abhyankar's Lemma in rigid geometry.

Theorem 2.2 (Rigid Abhyankar's Lemma)**.** *Let $N_1 = \mathrm{Sp}(A)$ be a smooth affinoid space over a p-adic field, let f_i be r functions which cut out r smooth divisors. Denote the union of these divisors by D. Let $N_2 \to N_1$ be a finite morphism which is étale away from D with $N_2 = \mathrm{Sp}(B)$ being normal. Then for sufficiently divisible $k \in \mathbb{N}$, the map*

$$A[\sqrt[k]{f_i}] \to (B \otimes_A A[\sqrt[k]{f_i}])^\nu$$

is finite étale.

Proof. Since the statement is local on N_1, we may assume (by [Kie67, Theorem 1.18], see also [Mit09, Theorem 2.11]) that $A = A_0\langle T_i \rangle$ where the divisor D is cut out by $T_1 T_2 \cdots T_r$. Then we see that $f_i = g_i \cdot T_i$ with g_i units.

For a $k \in \mathbb{N}$ to be chosen later, we let $X = \mathrm{Sp}(A\langle \sqrt[k]{f_i} \rangle)$, $Y = \mathrm{Sp}(A\langle \sqrt[k]{T_i} \rangle)$ and $W = (X \times_{N_1} Y)^\nu$. Note that $W \to X$ and $W \to Y$ are both finite étale, since they are given by adjoining k-th root of g_i.

$$\begin{array}{ccc} X & \longleftarrow & W \\ \downarrow & & \downarrow \\ N_1 & \longleftarrow & Y \end{array}$$

What we need to show is that after choosing k sufficiently divisible, the base change map

$$(N_2 \times_{N_1} X)^\nu \to X$$

is finite étale. But since $W \to X$ is finite étale, by descent it is enough to check after base changing to W. Because $W \to N_1$ also factors through Y, it suffices to choose k so that the base change to Y is étale. This can be achieved by Proposition 2.1. □

2.2. **Gysin sequence.** Let us gather facts concerning Gysin sequence (cohomological purity) in the setup of rigid spaces as developed by Berkovich (see [Ber95]) and Huber (c.f. [Hub96, Section 3.9]).

Theorem 2.3 (Gysin sequence)**.** *Let Y be a smooth rigid space over k, $\mathcal{E}$ an $\mathbb{F}_p$-local system on Y and $Z \subset Y$ a smooth divisor on Y. Then we have a long exact sequence*

$$H^2_{\text{ét}}(Y, \mathcal{E}) \to H^2_{\text{ét}}(Y \setminus Z, \mathcal{E}|_{Y \setminus Z}) \to H^1_{\text{ét}}(Z, \mathcal{E}|_Z(-1)) \to H^3_{\text{ét}}(Y, \mathcal{E}) \to \dots.$$

Here $\mathcal{E}|_Z(-1))$ means the Tate twist of the pullback $\mathcal{E}|_Z$ of $\mathcal{E}$ to Z.

Proof. This follows from a re-interpretation of [Ber95, 2.1 Theorem], where we apply the Theorem in loc. cit. to our case where S and (Y, X) from loc. cit. correspond to $\mathrm{Sp}(k)$ and (Z, Y), which satisfies the condition in loc. cit. □

2.3. **The site $X_{\log}$.** In this subsection we introduce the log-étale site $X_{\log}$ (also known as the Faltings site) of the pair (X, D) and show a comparison theorem between this site and $U_{\text{ét}}$. Note that this site depends on a choice of divisor D, however we suppress that in the notation for the sake of simplicity of notations.

Definition 2.4. Let f be a morphism between two objects $V_i \to X$ over X. We denote the restriction of f to V_2° by f°.

We define a site $X_{\log}$ as follows: an object of $X_{\log}$ consists of arrows

$$N \xrightarrow{f} V \xrightarrow{g} X$$

(denoted by (V, N)) such that

(1) the morphism g is étale;
(2) N is normal;
(3) the morphism f is finite with $f^\circ \colon N \backslash (g \circ f)^{-1}(D) \to V \backslash g^{-1}(D)$ being étale and;
(4) $(g \circ f)^{-1}(D)$ is nowhere dense in N.

A morphism in this site between (V, N) and (V', N') is given by a pair (p, q) of two maps in a commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{q} & N' \\ \downarrow & & \downarrow \\ V & \xrightarrow{p} & V'. \end{array}$$

The morphisms

$$\{(p_i, q_i) : (V_i, N_i) \to (V, N)\}$$

form a covering if $N = \bigcup q_i(N_i)$. Notice that by Lemma 2.6 (2) below, the image of N_i in N are open subsets.

Similarly, for a $V \to X$ étale over X we can define a subsite $V_{\mathrm{f,log}}$ whose objects are consisting of $N \xrightarrow{f} V$ satisfying condition 2.4(2)-(4). The morphisms are just usual morphisms in the category of rigid spaces over V. Note that by [Han17, Theorem 1.6], we have $V_{\mathrm{f,log}} \cong V^\circ_{\mathrm{f\acute{e}t}}$.

Remark 2.5. One should note the subtle difference between the above definition of the Faltings site and that in [AGT16, III.8.2]. In particular, the counterpart of the counterexample in [AGT16, III.8.18] in our the Faltings site here does not form a covering.

Lemma 2.6. *The category $X_{\log}$ has the following properties:*

(1) *finite projective limit and a terminal object exist. Moreover, the fiber products of $(V', N') \to (V, N)$ and $(V'', N'') \to (V, N)$ is given by*

$$\left(V' \times_V V'', (pr_1^* N' \times_{N|_W} pr_2^* N'')^\nu\right) = \left(W, (N' \times_N N'')^\nu\right)$$

where pr_i are the natural projections. In particular, the equalizer of two morphisms $(p, q), (s, t) : (V', N') \to (V, N)$ is given by $\left(eq(p, s), eq(q, t)^\nu\right)$ where $eq(\cdot, \cdot)$ is the equalizer of the two morphisms;
(2) *the image of the morphism $(V, N) \to (V', N')$ in $|N'|$ is open and;*
(3) *(V, N) is quasi-compact (resp. quasi-separated) if and only if N is quasi-compact (resp. quasi-separated) which will be valid if V is quasi-compact (resp. quasi-separated).*

Proof. Let us prove it for $X_{\log}$, the proof for $X_{\mathrm{f,log}}$ is similar.

Proof of (1). The existence of finite projective limit and the explicit descriptions just follow from [Han17, Theorem 1.6] and the existence and descriptions in $X_{\text{ét}}$ (for the V part) and $V^{\circ}_{f\text{ét}}$ (for the N part). The terminal object is clearly (X, X).

Proof of (2). Let us consider the morphism $N \to N'_V := N' \times_{V'} V$. We claim that the image is union of connected components of N'_V, (2) clearly follows from this claim. This claim follows from the fact that ${N'_V}^{\circ}$ has the same number of connected components as that of N'_V (see [Han17, Corollary 2.7]) and is dense in N'_V. But now since $N^{\circ} \to {N'_V}^{\circ}$ is finite étale, therefore the image is union of connected components of ${N'_V}^{\circ}$. Because $N \to N'_V$ is finite, therefore the image is closure of the image of corresponding circ map.

Proof of (3). Let us first show that if N is quasi-compact, then (V, N) is a quasi-compact object in this site. Let $(V_i, N_i) \to (V, N)$ be a covering. Because the image of N_i is a union of connected components of preimage of image of V_i, we see that it must be an open subset of N. Since N is quasi-compact, finitely many of $N_i \to N$ would have image covering N. Now if (V, N) is a quasi-compact object, it is obvious that the image of N in V is quasi-compact. Hence N being finite over that image, is also quasi-compact. The statement concerning quasi-separatedness just follows from the description of fibre product. The statement about V easily follows from the fact that $N \to V$ is finite. □

Definition 2.7. There is a natural morphism between sites

$$U_{\text{ét}} \to X_{\log},\ (V, N) \mapsto N^{\circ}$$

inducing a morphism between topoi

$$u_X : \mathrm{Sh}(U_{\text{ét}}) \to \mathrm{Sh}(X_{\log}).$$

The main result of this section is the following.

Theorem 2.8. *Let $\mathbb{L}$ be a $\mathbb{F}_p$-local system on U. Then we have*
(1) *$u_{X*}(\mathbb{L})(V, N) = \mathbb{L}(N^{\circ})$ for an object $(V, N) \in X_{\log}$ and;*
(2) *$R^i u_{X*}(\mathbb{L}) = 0$ for $i \geq 1$.*

Before proving this Theorem, let us state and prove some Lemmas.

Lemma 2.9. *Let $\mathcal{E}$ be a $\mathbb{F}_p$-local system on $S \times \mathbb{D}^{\circ,r}$ where S is a smooth connected affinoid space over k. Then there is a Kummer map $\mathbb{D}^r \xrightarrow{\varphi} \mathbb{D}^r$ (i.e. raise coordinates to sufficiently divisible power) such that $(id_S \times \varphi)^*(\mathcal{E})$ is a restriction of a $\mathbb{F}_p$-local system on $S \times \mathbb{D}^r$.*

Proof. It follows from Proposition 2.1 and the fact that $\mathcal{E}$ is represented by a finite étale covering of $S \times \mathbb{D}^{\circ,r}$. □

Lemma 2.10. *Let $\mathcal{E}$ be a $\mathbb{F}_p$-local system on $S \times \mathbb{D}^r \times \mathbb{D}^k = S \times \mathbb{D}^{r+k}$ where S is a smooth connected affinoid space over k. Then for every cohomology class $\alpha \in H^j_{\text{ét}}(S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k, \mathcal{E})$ where $j \geq 1$, there is a finite étale covering $N^{\circ} \xrightarrow{\Phi} S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k$ with $\Phi^*(\alpha) = 0$ in $H^j_{\text{ét}}(N^{\circ}, \Phi^*\mathcal{E})$.*

Proof. For $j = 1$, the lemma is easily deduced from the torsor interpretation of cohomology classes of degree 1.

In the following, we assume j is at least 2. We prove the lemma by the induction on r. When $r = 0$, it is a special case of [Sch13a, Theorem 4.9]. Suppose that the lemma holds for r. We consider the $(r + 1)$-case.

Note that

$$S \times \mathbb{D}^{\circ,r+1} \times \mathbb{D}^k = (S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^{k+1}) \setminus (S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k \times \{0\}) =: C \setminus \Delta_1$$

where $C = S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^{k+1}$ and $\Delta_1 = S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k \times \{0\}$. The Gysin sequence (Theorem 2.3) applied to the pair (C, Δ_1) gives the connecting map

$$H^j_{\text{ét}}(C \setminus \Delta_1, \mathcal{E}) \to H^{j-1}_{\text{ét}}(\Delta_1, \mathcal{E}|_{\Delta_1}(-1))$$

mapping α to β. By the induction and $j \geq 2$, there is a finite étale covering $f \colon \Delta'_1 \to \Delta_1$ with $f^*(\beta) = 0$. It give us a finite covering of C

$$f \times id_{\mathbb{D}} : \Delta'_1 \times \mathbb{D} \to \Delta_1 \times \mathbb{D} = S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k \times \mathbb{D} = C$$

whose restriction to $\Delta_1 \times \{0\}$ is f. The map $f \times id_{\mathbb{D}}$ induces a map from the Gysin sequence of $(\Delta'_1 \times \mathbb{D}, \Delta'_1)$ to that of (C, Δ_1) as follows:

$$\begin{array}{ccccccc}
H^j_{\text{ét}}(C) & \longrightarrow & H^j_{\text{ét}}(C \setminus \Delta_1) & \longrightarrow & H^{j-1}_{\text{ét}}(\Delta_1) & \longrightarrow & H^{j+1}_{\text{ét}}(C) \; . \\
\downarrow{\scriptstyle (f\times id_{\mathbb{D}})^*} & & \downarrow{\scriptstyle (f\times id_{\mathbb{D}})|^*_{C\setminus\Delta_1}} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f\times id_{\mathbb{D}})^*} \\
H^j_{\text{ét}}(\Delta'_1 \times \mathbb{D}) & \xrightarrow{h} & H^j_{\text{ét}}(\Delta'_1 \times \mathbb{D} \setminus \Delta'_1) & \longrightarrow & H^{j-1}_{\text{ét}}(\Delta'_1) & \longrightarrow & H^{j+1}_{\text{ét}}(C)
\end{array}$$

It follows that $(f \times id_{\mathbb{D}})|^*_{C\setminus\Delta_1}(\alpha) = h(\gamma)$ for some $\gamma \in H^j_{\text{ét}}(\Delta'_1 \times \mathbb{D}, \mathcal{E}|_{\Delta'_1 \times \mathbb{D}})$.

We claim that there is a finite étale covering $\theta \colon N \to \Delta'_1 \times \mathbb{D}$ with $\theta^*(\gamma) = 0$. This proves that

$$\left(\theta \circ (f \times id_{\mathbb{D}})|_{C\setminus\Delta_1}\right)^* (\alpha) = 0$$

which is what we need to show in the $(r+1)$-case.

Now we show the claim above. In fact, let $\mathcal{E}'$ be the $\mathbb{F}_p$-local system $(f \times id_{\mathbb{D}})_*(\mathcal{E}|_{\Delta'_1 \times \mathbb{D}})$ on C. Now γ can be viewed as an element in $H^j_{\text{ét}}(C, \mathcal{E}') = H^j_{\text{ét}}(\Delta'_1 \times \mathbb{D}, \mathcal{E}|_{\Delta'_1 \times \mathbb{D}})$. By Lemma 2.9, there is a finite étale covering

$$\varphi \colon S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^{k+1} \to S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^{k+1} = \Delta_1 \times \mathbb{D} = C$$

such that the pullback $\varphi^*(\mathcal{E}')$ is a restriction of a $\mathbb{F}_p$-local system on $S \times \mathbb{D}^r \times \mathbb{D}^{k+1}$. Therefore by the induction (applied to $\varphi^*(\gamma)$), we have a finite étale covering $g \colon W \to S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^{k+1}$ with $(\varphi \circ g)^*(\gamma) = g^*(\varphi^*(\gamma)) = 0$. Considering the Cartesian diagram,

$$\begin{array}{ccc}
N & \xrightarrow{\theta} & \Delta'_1 \times \mathbb{D} \\
\downarrow & & \downarrow{\scriptstyle f\times id_{\mathbb{D}}} \\
W & \xrightarrow{\varphi\circ g} & \Delta \times \mathbb{D} = C
\end{array}$$

we see that θ is a finite étale covering with $\theta^*(\gamma) = 0$. $\square$

Lemma 2.11. *Let $\mathcal{E}$ be a $\mathbb{F}_p$-local system on $S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k$ where S is a smooth connected affinoid space over k. Then for every cohomology class $\alpha \in H^j_{\text{ét}}(S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k, \mathcal{E})$ where $j \geq 1$, there is a finite étale covering $N^\circ \xrightarrow{\varphi} S \times \mathbb{D}^{\circ,r} \times \mathbb{D}^k$ with*

$$\varphi^*(\alpha) = 0 \quad in \quad H^j_{\text{ét}}(N^\circ, \varphi^*\mathcal{E}).$$

Proof. It follows from Lemma 2.9 and Lemma 2.10. $\square$

Proposition 2.12. *Let S be a smooth connected affinoid space over K, and let $\mathbb{D}^r$ be the unit ball with coordinates $z_1, \dots, z_r$. Set $\Delta = V(z_1 \cdots z_r)$ and $\mathbb{D}^{\circ,r} = \mathbb{D}^r - \Delta$. Let $f : N^\circ \to S \times \mathbb{D}^{\circ,r}$ be a finite étale covering. For a $\mathbb{F}_p$-local system $\mathbb{L}$ on $S \times \mathbb{D}^{\circ,r}$ and a cohomology class $\alpha \in H^i_{\text{ét}}(N^\circ, \mathbb{L}|_{N^\circ})$ where $i \geq 1$, there is a finite étale covering $\varphi : M^\circ \to N^\circ$ such that*

$$\varphi^*(\alpha) = 0 \in H^i_{\text{ét}}(M^\circ, \mathbb{L}|_{M^\circ}).$$

Proof. This follows from applying Lemma 2.11 to $\mathcal{E} = f_*(\mathbb{L})$ and $f_*(\alpha)$. □

Now we are ready to give the

Proof of Theorem 2.8. The statement (1) is obvious. It is clear that $R^i u_{X*}(\mathbb{L})$ is the sheaf associated to the presheaf

$$(N \xrightarrow{f} V \xrightarrow{g} X) \to H^i_{\text{ét}}(N^\circ, (g \circ f)^*\mathbb{L}) = H^i_{\text{ét}}(N^\circ, \mathbb{L}|_{N^\circ}).$$

The statement (2) is a local property, hence (by [Kie67, Theorem 1.18]) we may assume that V is $S \times \mathbb{D}^r$ with finite

$$f : N \to S \times \mathbb{D}^r = V$$

such that f° is étale and $V^\circ = S \times \mathbb{D}^{\circ,r}$ where S is a smooth and connected affinoid space over k. By [Han17, Theorem 1.6] it suffices to show that, for every cohomology class $\alpha \in H^i_{\text{ét}}(N^\circ, \mathbb{L}|_{N^\circ})$, there is a finite étale covering $N'^\circ \xrightarrow{g} N^\circ$ such that $g^*(\alpha) = 0$. But this follows from Proposition 2.12. □

3. The site X_{prolog}

In this section we introduce the pro-log-étale site X_{prolog} of the pair (X, D) and show a comparison theorem between it and the previous site X_{log}. It is parallel to [Sch13a, Section 3] except we will use a categorical way to introduce this site.

In the following, we denote by $\mathcal{C}$ a category which has arbitrary finite projective limits and a distinguished terminal object X.

Let $\mathcal{C}_f$ be a wide (i.e. lluf) subcategory of $\mathcal{C}$ such that the morphisms of $\mathcal{C}_f$ are stable under the base change via any morphism in $\mathcal{C}$, i.e. if $W \to V \in \text{Hom}_{\mathcal{C}_f}$, then $W \times_V Z \to Z$ is in $\text{Hom}_{\mathcal{C}_f}$ for any $Z \to V$. For the category $\mathcal{C}$, we have a functor $|-|_{\mathcal{C}} : \mathcal{C} \to Top$ from $\mathcal{C}$ to the category of topological spaces such that

$$|A \times_B C|_{\mathcal{C}} \to |A|_{\mathcal{C}} \times_{|B|_{\mathcal{C}}} |C|_{\mathcal{C}}$$

is surjective with finite fibers for any maps $A \to B$ and $C \to B$ in $\mathcal{C}$. Consider the pro-category pro-$\mathcal{C}$ of $\mathcal{C}$. The functor $|-|_{\mathcal{C}}$ extends to a functor from pro-$\mathcal{C}$ to Top by $|\varprojlim N_i| = \varprojlim |N_i|_{\mathcal{C}}$, and we denote it by $|-|$.

In the category of pro-$\mathcal{C}$, we can define several types of morphisms.

Definition 3.1. Let $W \to V$ be a morphism of pro-$\mathcal{C}$. We say $W \to V$ is a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map) if $W \to V$ is induced by a morphism $W_0 \to V_0$ in $\mathcal{C}$ (resp. $\mathcal{C}_f$), i.e. $W = V \times_{V_0} W_0$ via some map $V \to V_0$.

We say that $W \to V$ is surjective if the corresponding map $|W| \to |V|$ is surjective. We say $W \to V$ is a pro-$\mathcal{C}$ map if $W = \varprojlim W_j$ can be written as a cofiltered inverse limit of $\mathcal{C}$ maps $W_i \to V$ over V and $W_j \to W_i$ are surjective C_f maps for large $j > i$. Note that W_i is an object of pro-$\mathcal{C}$. The presentation $W = \varprojlim W_i$ is called a pro-$\mathcal{C}$ presentation.

We define a full subcategory $X_{\mathrm{pro}\mathcal{C}}$ of pro-$\mathcal{C}$. The object of this category consists of objects in pro-$\mathcal{C}$ which are pro-$\mathcal{C}$ over X, i.e. each object has a pro-$\mathcal{C}$ map to X. The morphisms are pro-$\mathcal{C}$ maps.

The following lemma is almost identical to [Sch13a, Lemma 3.10], except we do not state the seventh sub-statement (which is the only non-formal one) here.

Lemma 3.2.

(1) *Let $W \to V$ be a surjective morphism in $\mathcal{C}$. For any morphism $W' \to V$ in $\mathcal{C}$, the base change $W' \times_V W \to W'$ is surjective.*
(2) *Let $W \to V$ be a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map, resp. pro-$\mathcal{C}$ map) in pro-$\mathcal{C}$. For any morphism $W' \to V$ in pro-$\mathcal{C}$, the base change $W' \times_V W \to W'$ is a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map, resp. pro-$\mathcal{C}$ map) and the map $|W' \times_V W| \to |W'| \times_{|V|} |W|$ is surjective, in particular, $W' \times_V W \to W'$ is surjective if $W' \to V$ is.*
(3) *A composition of $E \to F \to G$ of two $\mathcal{C}$ maps (resp. $\mathcal{C}_f$ maps) in pro-$\mathcal{C}$ is a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map).*
(4) *A surjective $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map) $W \to V$ with $V \in X_{\mathrm{pro}\mathcal{C}}$ comes from a pull back via $V \to V_0$ from a surjective map $W_0 \to V_0$ with $W_0, V_0 \in \mathcal{C}$.*
(5) *Let $E \to F \to G \to X$ be a sequence of morphisms where all the arrows are pro-$\mathcal{C}$ maps. Then $E, F \in X_{\mathrm{pro}\mathcal{C}}$ and the composition $E \to G$ is a pro-$\mathcal{C}$ map.*
(6) *If all maps in $\mathcal{C}$ have open images, then any pro-$\mathcal{C}$ map $W \to V$ in pro-$\mathcal{C}$ has open image.*

Proof.

(1) It follows from the surjectivity of the map $|W' \times_V W|_{\mathcal{C}} \to |W'|_{\mathcal{C}} \times_{|V|_{\mathcal{C}}} |W|_{\mathcal{C}}$.
(2) If $W \to V$ is a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map) then by definition we reduce to the case $W, V \in \mathcal{C}$. Write $W' = \varprojlim W'_i$ with a compatible system of maps $W'_i \to V \in \mathcal{C}$. Then $W \times_V W' = \varprojlim W \times_V W'_i$ and $W \times_V W' \to W'$ is by definition again a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map). As for the topological spaces, we have

$$|W' \times_V W| = \varprojlim |W \times_V W'_i| \to \varprojlim |W| \times_{|V|} |W'_i| = |W| \times_{|V|} |W'|$$

where the first equality follows from definition, and the last equality is due to that fiber products commute with inverse limits. The middle map is surjective because it is surjective with compact fibers at each finite stage, and inverse limits of nonempty compact spaces are nonempty. Actually, the fibers are nonempty compact spaces.

In the general case, take a pro-$\mathcal{C}$ presentation $W' = \varprojlim W'_i \to V$. Then we have that $W' \times_V W = \varprojlim W'_i \times_V W \to W'$ is a pro-$\mathcal{C}$ map over W' by what we have just proved. The map

$$|W' \times_V W| = \varprojlim |W \times_V W'_i| \to \varprojlim |W| \times_{|V|} |W'_i| = |W| \times_{|V|} |W'|$$

is surjective by the same reasoning as before.
(3) Write $F = F_0 \times_{G_0} G$ as a pullback of a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map) $F_0 \to G_0$. Moreover, write $G = \varprojlim G_i$ with a compatible system of maps $G_i \to G_0 \in \mathcal{C}$. Then $F = \varprojlim (G_j \times_{G_0} F_0)$.

Moreover write $E = F \times_{F'_0} E_0$ as a pullback of a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map) $E_0 \to F'_0$. Therefore the map $F \to F'_0$ factors through $G_j \times_{G_0} F_0 \to F'_0$ for large j. It follows that

$$E = F \times_{(G_j \times_{G_0} F_0)} \left((G_j \times_{G_0} F_0) \times_{F_0'} E_0 \right)$$

and $E \to G$ is a pullback of a $\mathcal{C}$ map (resp. $\mathcal{C}_f$ map).

(4) Let $V = \varprojlim V_i$ be a pro-$\mathcal{C}$ presentation over X. Note that $W \to V$ is induced by a pullback of a morphism $W_0 \to V_0$ with $W_0, V_0 \in \mathcal{C}$ via some map $V \to V_0$. The map $V \to V_0$ factors throguh a map $V_j \to V_0$ for large j. Therefore, we have $W = W_0 \times_{V_0} V = (W_0 \times_{V_0} V_j) \times_{V_j} V$. On the other hand, $|V| \to |V_j|$ is surjective for large j. Therefore $W_0 \times_{V_0} V_j \to V_j$ is surjective.

(5) One can write $E \to F$ as the composition $E \to E_0 \to F$ of an inverse system $E = \varprojlim E_i \to E_0$ of surjective $\mathcal{C}_f$ maps $E_i \to E_j \to E_0$, and a $\mathcal{C}$ map $E_0 \to F$. We check the statement separately in the case that $E \to F$ is a $\mathcal{C}$ map or an inverse system of surjective $\mathcal{C}_f$ maps. Assume that $E \to F$ is a $\mathcal{C}$ map which is induced by a map $E_0 \to F_0 \in \mathcal{C}$, i.e. $E = F \times_{F_0} E_0$ via some map $F \to F_0$. Write $F = \varprojlim F_i \to G$ as a pro-C presentation. Therefore, $F \to F_0$ factors through $F_i \to F_0$ for large i. It follows from (2) and (3) that $E = F \times_{F_0} E_0 = \varprojlim (F_i \times_{F_0} E_0) \to G$ is a pro-$\mathcal{C}$ presentation over G, in other words, the composition $E \to G$ is a pro-$\mathcal{C}$ map.

So it reduces us to consider all maps $E \to F \to G \to X$ are inverse systems of surjective $\mathcal{C}_f$ maps. Using (1) and (4), it is an easy exercise to show that all the compositions are still inverse systems of surjective $\mathcal{C}_f$ maps.

(6) Let $U \to V$ be a pro-$\mathcal{C}$ map with a pro-$\mathcal{C}$ presentation $U = \varprojlim U_i \to V$. Therefore we have $|U| \to |U_i| \to |V|$ with $|U| \to |U_i|$ surjective. It reduces us to show $|U_i| \to |V|$ has open image. Since $U_i \to V$ is a C map, we have $U_i = V \times_{V_0} U_{i0}$ for some map $U_{i0} \to V_0$ in $\mathcal{C}$. By (2), we have a surjection $|U_i| = |V \times_{V_0} U_{i0}| \to |V| \times_{|V_0|} |U_{i0}|$. Therefore, the image of $|U_i| \to |V|$ is open.

□

We declare coverings in $X_{\mathrm{pro}\mathcal{C}}$ as following: a covering in $X_{\mathrm{pro}\mathcal{C}}$ is given by a family of pro-$\mathcal{C}$ maps $\{f_i \colon V_i \to V\}$ such that $|V| = \bigcup_i |f_i|(|V_i|)$. From Lemma 3.2, we know that $X_{\mathrm{pro}\mathcal{C}}$ is a site.

Lemma 3.3. *Let $M \to N$ be a pro-$\mathcal{C}$ map. If $M \to N$ is a covering of $X_{\mathrm{pro}\mathcal{C}}$, then $M \to N$ is induced by a covering $M_0 \to N_0$ of $\mathcal{C}$.*

Proof. Write $N = \varprojlim N_i$ as a pro-$\mathcal{C}$ presentation of N. It follow that $|N| \to |N_i|$ is surjective for large i. Note that $M \to N$ is induced by a morphism $M_0' \to N_0'$ in $\mathcal{C}$ via some map $N \to N_0'$. The map $N \to N_0'$ factors over $N_i \to N_0'$ for large i. Therefore, the map $M \to N$ is induced by the map $N_i \times_{N_0'} M_0' \to N_i$. Hence $N_i \times_{N_0'} M_0' \to N_i$ is a covering, and we may choose N_i (resp. $N_i \times_{N_0'} M_0'$) to be the N_0 (resp. M_0) we are looking for. □

Example 3.4. We can take $\mathcal{C} = X_{\text{ét}}$ and $\mathcal{C}_f$ to be the wide subcategory only allowing finite étale maps to be morphisms. The functor $| - |_{\mathcal{C}}$ is the functor associating to an object its underlying topological space. Then $X_{\mathrm{pro}\mathcal{C}}$ is just the pro-étale site $X_{\text{proét}}$ introduced in [Sch13a].

Example 3.5. Now we specialize our construction above to the Faltings site $\mathcal{C} = X_{\log}$, $|(V,N)|_{\mathcal{C}} = |N|$ for $(V,N) \in X_{\log}$ and we take $\mathcal{C}_f$ to be the wide subcategory only allowing morphisms of the form $(V,N) \to (V,N')$, namely, every morphism of $\mathcal{C}_f$ is a morphism in $V_{\mathrm{f,log}} \cong V^{\circ}_{\mathrm{fét}}$ for some $V \in X_{\text{ét}}$. Recall that a fiber product of morphisms $(V',N') \to (V,N)$ and $(V'',N'') \to (V,N)$ in $X_{\log}$ is given by

$$\left(W = V' \times_V V'', (pr_1^* N' \times_{N \times_V W} pr_2^* N'')^{\nu} \right) \tag{3.6}$$

where $pr_1 : W \to V'$ and $pr_2 : W \to V''$ are the natural projections. It is easy to check $\mathcal{C}, \mathcal{C}_f$ and $|-|_{\mathcal{C}}$ satisfy our assumptions of previous results, hence produce a site X_{prolog}. In this site, we will call a pro-$\mathcal{C}$ map (resp. $\mathcal{C}$ map, $\mathcal{C}_f$ map, pro-$\mathcal{C}$ presentation) by pro-log-étale map (resp. log étale map, finite log étale map, pro-log-étale presentation).

In concrete terms, the objects of X_{prolog} are of the form (V,N) where $N = \varprojlim N_i$ is a tower of $N_i \xrightarrow{f_i} V$ such that f_i is finite with f_i° étale and $N_i \to N_j$ is surjective for large $i > j$. The space $|(V,N)|$ is given by $\varprojlim |(V,N_i)| = \varprojlim |N_i|$. The category X_{prolog} has a natural fibered category structure over $X_{\text{ét}}$, namely we have a natural functor $X_{\mathrm{prolog}} \to X_{\text{ét}}$ sending (V,N) to V, and associating a morphism $V \xrightarrow{p} V'$ in $X_{\text{ét}}$ the pullback map sending $N' = \varprojlim N'_i$ to $p^*(N') = \varprojlim p^*(N'_i) = \varprojlim (N'_i \times_{V'} V)$.

If there is no confusion seemingly to arise, we will denote an object $(V,N) \in X_{\mathrm{prolog}}$ by N.

Lemma 3.7.

(1) *The category X_{prolog} has arbitrary finite projective limits.*
(2) *We have $\pi((V',N') \times_{(V,N)} (V'',N')) = V' \times_V V''$ where π is the fibered structure functor $X_{\mathrm{prolog}} \xrightarrow{\pi} X_{\text{ét}}$.*
(3) *The pro-log-étale morphisms in pro-$X_{\log}$ have open images.*

Proof.

(1) It suffices to check that finite products and equalizers exist. The first case follows from Lemma 3.2 which is formal. The non-formal prat is to check for equalizers and we need to use the fact that locally $|N|$ has only a finite number of connected components. In fact, suppose that $f, g : N' \to N$ are two morphism of X_{prolog}. By (the proof of) [KS06, Corollary 6.1.8], we can write $N' = \varprojlim N'_i$ and $N = \varprojlim N_i$ as pro-log-étale presentations with the same index category and maps $f_i, g_i : N'_i \to N_i$ such that $f = \varprojlim f_i$ and $g = \varprojlim g_i$. Let E_i be the equalizer of f_i and g_i in $X_{\log}$. We get the following diagram (cf. Lemma 2.6):

$$\begin{array}{ccccc} E_i & \longrightarrow & N'_i & \overset{f_i}{\underset{g_i}{\rightrightarrows}} & N_i \\ \downarrow & & \downarrow & & \downarrow \\ E_j & \longrightarrow & N'_j & \overset{f_j}{\underset{g_j}{\rightrightarrows}} & N_i \end{array}$$

where $N'_i \to N'_j$ and $N_i \to N_j$ are surjective for large i. We may assume that V_i and V'_i are affinoids. Denote the image of E_i in N'_j by E^i_j. Note that E_j^i is open and closed by Lemma 2.6 (2). Since N'_j has finitely many connected components, the image E_j^i stabilizes for i larger than some i_j. Hence we see $E_j^\infty = E_j^{i_j}$ is the equalizer that we are looking for.

(2) This follows from the description of fiber product in $X_{\log}$.

(3) It follows from Lemma 2.6 (2) and Lemma 3.2 (6).

□

Lemma 3.8.

(1) *For an object $(V, N) \in X_{\text{prolog}}$, if V is affinoid, then (V, N) is a quasi-compact object of X_{prolog}.*

(2) *The family of all objects $(V, N) \in X_{\text{prolog}}$ with V affinoid is generating X_{prolog}, and stable under fiber products.*

(3) *The topos $\mathrm{Sh}(X_{\text{prolog}})$ is algebraic and all objects (V, N) of X_{prolog} with V affinoid are quasi-compact and quasi-separated.*

(4) *An object $(V, N) \in X_{\text{prolog}}$ is quasi-compact if and only if $|(V, N)|$ is quasi-compact.*

(5) *An object $(V, N) \in X_{\text{prolog}}$ is quasi-separated if and only if $|(V, N)|$ is quasi-separated.*

Proof.

(1) It follows from Lemma 3.7 (3) that an object (W, M) of X_{prolog} is quasi-compact if $|(W, M)|$ is quasi-compact. If V is affinoid, we can write $N = \varprojlim N_i$ with N_i affinoid. Moreover, the space $|N_i|$ is a spectral space and the transition maps are spectral. Hence the inverse limit $\varprojlim |N_i|$ is a spectral space, and in particular quasi-compact. It follows that (V, N) is a quasi-compact object of X_{prolog}.

(2) For an object $(V, N) \in X_{\text{prolog}}$, we can use affinoid objects to cover V, i.e., $V = \cup V_i$. It is clear that $\{(V_i, N|_{V_i})\}$ is a covering of (V, N) in X_{prolog}. The family is obviously stable under fiber products.

(3) By (2) and [SGA72, VI Proposition 2.1], the topos $\mathrm{Sh}(X_{\text{prolog}})$ is locally algebraic (see [SGA72, VI Definition 2.3]) and all objects (V, N) of X_{prolog} with V affinoid are quasi-compact and quasi-separated. We check the criterion of [SGA72, VI Proposition 2.2 (ii bis)] by considering the class of (V, N) as in (1) that $V \to X$ factors over an affinoid open subset V_0 of X. It consists of coherent objects and is still generating X_{prolog}. Note that $(V, N) \times_{(X,X)} (V, N) = (V, N) \times_{(V_0,V_0)} (V, N)$ is an object as in (1) which is quasi-separated.

(4) Without loss of generality we may assume $|N| \to |V|$ is surjective. Therefore, the space $|V|$ is quasi-compact if $|(V, N)|$ is quasi-compact. Use finitely many objects (V_i, N_i) of form in (1) to cover (V, N). Note that (V_i, N_i) are quasi-compact by (1). It follows that (V, N) is quasi-compact. Conversely, if (V, N) is compact, then we can find finitely many (V_i, N_i) with V_i affinoid cover V. Note that $|(V_i, N_i)|$ is quasi-compact by the proof of (1). It follows that $|(V, N)|$ is quasi-compact.

(5) Cover (V, N) by $(V_i, N|_{V_i})$ as in the proof of (2). It follows from [SGA72, VI Colloary 1.17] that the object (V, N) is quasi-separated if and only if $(V_i, N|_{V_i}) \times_{(V,N)} (V_j, N|_{V_j})$ is quasi-compact if and only if $|(V_i, N|_{V_i})| \times_{|(V,N)|} |(V_j, N|_{V_j})|$ is quasi-compact if and only if $|(V, N)|$ is quasi-separated.

□

There is a natural projection $\nu : \mathrm{Sh}(X_{\mathrm{prolog}}) \to \mathrm{Sh}(X_{\log})$ induced by the morphism of sites $X_{\mathrm{prolog}} \to X_{\log}$ sending (V, N) to the constant tower $(V, \varprojlim N)$.

Lemma 3.9.

(1) *Let $\mathcal{F}$ be an abelian sheaf on $X_{\log}$. For any quasi-compact and quasi-separated $(V, N) = (V, \varprojlim N_j) \in X_{\mathrm{prolog}}$ and any $i \geq 0$, we have*

$$H^i((V, N), \nu^*\mathcal{F}) = \varinjlim H^i((V, N_j), \mathcal{F}).$$

(2) *Let $\mathcal{F}$ be an abelian sheaf on $X_{\log}$. The adjunction morphism $\mathcal{F} \to R\nu_*\nu^*(\mathcal{F})$ is an isomorphism.*

Proof. (1) We may assume that $\mathcal{F}$ is injective and that X is quasi-compact and quasi-separated. Let us work with the subsite $X_{\mathrm{prolog}qc} \subset X_{\mathrm{prolog}}$ consisting of quasi-compact objects; note that $\mathrm{Sh}(X_{\mathrm{prolog}qc}) = \mathrm{Sh}(X_{\mathrm{prolog}})$. Define a presheaf $G((V, N)) = \varinjlim \mathcal{F}((V, N_i))$ where $N = \varprojlim N_i$ with $N_i \in V_{\mathrm{f,log}}$. It is clear that $\nu^*\mathcal{F}$ is the sheaf associated to G. It suffices to show G is a sheaf with $H^i((V, N), G) = 0$ for all $(V, N) \in X_{\mathrm{prolog}qc}$ and $i > 0$. By [SGA72, V Proposition 4.3 (i) and (iii)], we just need to prove that for any $(V, N) \in X_{\mathrm{prolog}qc}$ with a pro-log-étale covering $(V_k, N_k) \to (V, N)$ in $X_{\mathrm{prolog}qc}$, the correspondding Cech complex

$$0 \to G((V, N)) \to \prod_k G((V_k, N_k)) \to \prod_{k,k'} G((V_k, N_k) \times_{(V_k, N_k)} (V_{k'}, N_{k'})) \to \dots$$

is exact. This shows that G is a sheaf and then all higher cohomology groups vanish.

Since (V, N) is quasi-compact, we can pass to a finite subcover and combine them into a single morphism $(V', N') \xrightarrow{(p,q)} (V, N)$. Write it in a pro-log-étale presentation $N' = \varprojlim N'_i \to N$. In the following, we write the Cech complex of G with respect to the covering (p, q) as $\mathrm{Cech}(N' \to N)$. Therefore, we have

$$\mathrm{Cech}(N' \to N) = \varinjlim \mathrm{Cech}(N'_i \to N)$$

where $N'_i \to N$ is a covering for large i. Therefore, it suffices to show the exactness of $\mathrm{Cech}(N'_i \to N)$. By Lemma 3.3, the cover $N'_i \to N$ is induced by a cover $N'_0 \to N_0$ in $X_{\log}$, i.e. $N'_i = N'_0 \times_{N_0} N$. Therefore, $\mathrm{Cech}(N'_i \to N)$ is the direct limit of the Cech complexes for some covers in $X_{\log}$. But this is acyclic by the injectivity of G on $X_{\log}$.

(2) Note that $R^i\nu_*\nu^*\mathcal{F}$ is the sheaf on $X_{\log}$ associated to the presheaf $(V, N) \mapsto H^i((V, N), \nu^*\mathcal{F})$ where (V, N) is considered as an element of X_{prolog}. Hence, (1) says that we get an isomorphism for $i = 0$. Moreover, for i positive, (1) says that $H^i((V, N), \nu^*\mathcal{F}) = H^i((V, N), \mathcal{F})$ if $(V, N) \in X_{\log}$ is quasi-compact and quasi-separated. By the local acyclicity of higher cohomology group, a section of $H^i((V, N), \mathcal{F})$ vanishes locally in the topology $X_{\log}$, so the associated sheaf is trivial. It follows that $R^i\nu_*\nu^*\mathcal{F} = 0$ for $i > 0$.

□

4. The structure sheaves

In this section, parallel to [Sch13a, Section 4], we introduce the structure sheaves $\mathcal{O}^+$, $\mathcal{O}$, $\hat{\mathcal{O}}^+$ and $\hat{\mathcal{O}}$ on our site X_{prolog}. In the following we will not distinguish rigid spaces and their associated adic spaces.

Definition 4.1. With the notations as in Definition 2.7, let X be a rigid space over $\mathrm{Sp}(k)$ with an SSNC divisor D. Consider the following sheaves on $X_{\log}$ and X_{prolog}.

(1) The integral structure sheaf $\mathcal{O}^+_{X_{\log}}$ on $X_{\log}$ is given by $u_{X,*}(\mathcal{O}^+_{U_{\text{ét}}})$. By [Han17, Theorem 2.6] we have $\mathcal{O}^+_{X_{\log}}((V,N)) = \mathcal{O}^+_N(N)$ for an object $(V,N) \in X_{\log}$. The structure sheaf $\mathcal{O}_{X_{\log}}$ on $X_{\log}$ is given by $\mathcal{O}^+_{X_{\log}}[\frac{1}{p}] = u_{X,*}(\mathcal{O}^+_{U_{\text{ét}}})[\frac{1}{p}]$, namely, $\mathcal{O}_{X_{\log}}(V,N) = \mathcal{O}_N(N)$ for quasi-compact and quasi-separated (V,N).
(2) The (uncompleted) structure sheaf is defined to be $\nu^*\mathcal{O}_{X_{\log}}$ on X_{prolog} with subring of integral elements $\nu^*\mathcal{O}^+_{X_{\log}}$. If no confusion seems to arise, we will still denote them by $\mathcal{O}_{X_{\log}}$ and $\mathcal{O}^+_{X_{\log}}$ respectively.
(3) We define the completed integral structure sheaf (on X_{prolog}) to be $\hat{\mathcal{O}}^+_{X_{\log}} = \varprojlim \mathcal{O}^+_{X_{\log}}/p^n$, and the completed structure is defined as $\hat{\mathcal{O}}_{X_{\log}} = \hat{\mathcal{O}}^+_{X_{\log}}[\frac{1}{p}]$.

For simplicity, for the rest of this section we assume X is a rigid space over a perfectoid field K.

Definition 4.2. Let $(V,N) \in X_{\text{prolog}}$ with $V \xrightarrow{f} X$. We say that (V,N) is affinoid perfectoid if

(1) V is affinoid with $V = \mathrm{Sp}(R')$ and $f^{-1}(D_i)$ is cut out by one equation f_i;
(2) N has a presentation $N = \varprojlim N_i$ for a cofiltered system $\{N_i = \mathrm{Sp}(R_i)\}$ of objects in $V_{\text{f,log}}$ such that
 - N_i are smooth;
 - $\{N_i\}$ contains a cofiltered subsystem consisting of all branched coverings $\mathrm{Sp}\left(R'[\sqrt[k]{f_i}]\right)$ for all $k \in \mathbb{N}$ and;
 - denote by R^+ the p-adic completion of $\varinjlim R_i^\circ$, and $R = R^+[\frac{1}{p}]$, the pair (R, R^+) is a perfectoid affinoid (K, K°)-algebra.

Remark 4.3. In the above definition (2), one can actually drop the first condition. Indeed, any cofiltered system satisfying second condition automatically has a cofinal subsystem with N_i being smooth by Theorem 2.2.

We say that (V,N) is perfectoid if it has an open cover by affinoid perfectoid. To an affinoid perfectoid (V,N) as above, we can associate $\hat{N} = \mathrm{Spa}(R, R^+)$ which is an affinoid perfectoid space over $\mathrm{Spa}(K, \mathcal{O}_K)$. One immediately checks that this is well-defined, i.e. independent of the presentation of $N = \varprojlim N_i$. Moreover, we have $\hat{N} \sim \varprojlim N_i$ in the sense of [Sch12, Definition 7.14], in particular $|\hat{N}| = |N|$.

Example 4.4. Take

$$X = V = \mathrm{Sp}\left(K\langle Z_1^{\pm 1}, \dots, Z_{n-r}^{\pm 1}, Z_{n-r+1}, \dots, Z_n\rangle\right) = \mathbb{T}^{n-r} \times \mathbb{D}^r,$$

denote it by $\mathbb{T}^{n-r,r}$, with the divisor D given by $Z_{n-r+1} \cdots Z_n = 0$. Then $(\mathbb{T}^{n-r,r}, N) \in X_{\text{prolog}}$ with $N = \tilde{\mathbb{T}}^{n-r,r}$ being the inverse limit of the

$$\mathrm{Sp}\left(K\langle Z_1^{\pm 1/p^k}, \dots, Z_{n-r}^{\pm 1/p^k}, Z_{n-r+1}^{1/l}, \dots, Z_n^{1/l}\rangle\right)$$

is an affinoid perfectoid. Using the notations from discussion before this example, we have

$$R = K\langle Z_1^{\pm 1/p^\infty}, \dots, Z_{n-r}^{\pm 1/p^\infty}, Z_{n-r+1}^{1/\infty}, \dots, Z_n^{1/\infty}\rangle$$

and

$$R^+ = \mathcal{O}_K\langle Z_1^{\pm 1/p^\infty}, \dots, Z_{n-r}^{\pm 1/p^\infty}, Z_{n-r+1}^{1/\infty}, \dots, Z_n^{1/\infty}\rangle.$$

The following lemma is an analogue of [Sch13a, Lemma 4.5]. The proof is exactly the same.

Lemma 4.5. *With the notations as in Definition 4.2, let $(V, N) \in X_{\text{prolog}}$ be an affinoid perfectoid with $N = \varprojlim N_i$ and $N_i = \operatorname{Spa}(R_i, R_i^\circ)$ so that $\hat{N} = \operatorname{Spa}(R, R^+)$.*

Assume that $M_i = \operatorname{Spa}(S_i, S_i^\circ) \to N_i$ is an étale map which can be written as a composition of rational subsets and finite étale maps. For $j \geq i$, write $M_j = M_i \times_{N_i} N_j = \operatorname{Spa}(S_j, S_j^\circ)$ and $M = M_i \times_{N_i} N = \varinjlim M_j \in$ pro-(Rigid/M_i). Let A_j be the p-adic completion of the p-torsion free quotient of $S_j^\circ \otimes_{R_j^\circ} R^+$. Then

1. *The completion (S, S^+) of the direct limit of the (S_j, S_j°) is a perfectoid affinoid (K, K°)-algebra. Moreover, $\hat{M} = M_j \times_{M_j} \hat{N}$ in the category of adic spaces over $\operatorname{Spa}(K, K^\circ)$, and $S = A_j[\frac{1}{p}]$ for any $j \geq i$, where $\hat{M}$ is similarly defined as $\hat{N}$, i.e. $\hat{M} = \operatorname{Spa}\left((\hat{\varinjlim} S_j^\circ)[\frac{1}{p}], \hat{\varinjlim} S_j^\circ\right)$.*
2. *For any $j \geq i$, the cokernel of the map $A_j \to S^+$ is annihilated by some power p^N of p.*
3. *Let $\epsilon > 0$, $\epsilon \in \log \Gamma$. Then there exists some j such that the cokernel of the map $A_j \to S^+$ is annihilated by p^ϵ.*

Proof. The proof is the same as [Sch13a, Lemma 4.5]. Roughly speaking, for $M_i \subseteq N_i$ being a rational subset, it follows from the property that a rational subset of an affinoid perfectoid space is affinoid perfectoid, see [Sch12, Theorem 6.3 (ii)]. For $M_i \subseteq N_i$ being a finite étale morphism, it follows from the almost purity theorem [Sch12, Theorem 7.9(iii)]. □

Lemma 4.6. *Let $(V, N') \to (V, N)$ be a finite log étale morphism in X_{prolog}. If (V, N) is affinoid perfectoid, then the morphism $(V, N') \to (V, N)$ is induced by a finite étale morphism between two objects of $V_{\text{f,log}}$, i.e. $N' = N \times_{N_0} N_0'$ via some finite étale morphism $N_0' \to N_0$, and (V, N') is affinoid perfectoid.*

Proof. Use the notations as in Definition 4.2. Suppose that $(V, N') \to (V, N)$ is induced by $(V, N_0') \to (V, N_0)$ in $V_{\text{f,log}}$, i.e. $(V, N') = (V, N_0') \times_{(V, N_0)} (V, N)$ via some map $N \to N_0$ of pro-$V_{\text{f,log}}$ where N_0 is smooth. By Lemma 2.2, we know there is $N_0[\sqrt[k]{f_i}] = N_1 \to N_0$ for large k such that $N_1' \coloneqq N_1 \times_{N_0} N_0' \to N_1$ is finite étale. Now by our assumption of (V, N) being affinoid perfectoid, we may find N_2 inside the tower of N dominating N_1. Therefore N' is induced by the morphism $N_2' \coloneqq N_2 \times_{N_0} N_0' \to N_2$ which is finite étale. One checks (V, N') is an affinoid perfectoid: it consists of cofinal system of smooth N_j''s since N' is induced by a finite étale morphism; the completed algebra being perfectoid follows from almost purity (see [Sch12, Theorem 7.9]); and since our system has a subsystem dominating $\operatorname{Sp}(R'[\sqrt[k]{f_i}])$, throwing them in our system gives rise to a presentation of N'. □

Theorem 4.7. *The set of $(V, N) \in X_{\text{prolog}}$ which are affinoid perfectoid form a basis for the topology.*

Proof. Use the notations as in Example 4.4. If $(X, D) = (\mathbb{T}^{n-r,r}, D)$, then we have made an explicit cover of X by an affinoid perfectoid $\tilde{\mathbb{T}}^{n-r,r} \in X_{\text{prolog}}$. Let (V, N) be an object of X_{prolog} with $V \to X$ étale, $N = \varprojlim N_i \xrightarrow{h} V$ where $N_i \xrightarrow{h_i} V \in V_{\text{f,log}}$. By [Kie67, Theorem 1.18] and [Hub96, Corollary 1.6.10], we may assume that V admits an étale morphism $V \xrightarrow{f} \mathbb{T}^{n-r,r}$ with divisor given by $f^{-1}(D)$. We may further assume that f is the composite of a rational open embedding and a finite étale morphism. Therefore, $(V, f^*(\tilde{\mathbb{T}}^{n-r,r})) = (V, V \times_{\mathbb{T}^{n-r,r}} \tilde{\mathbb{T}}^{n-r,r}) \in X_{\text{prolog}}$ is affinoid perfectoid by Lemma 4.5. By Lemma 4.6, we know $(V, h_i^*(f^*(\tilde{\mathbb{T}}^{n-r,r}))$ is also affinoid perfectoid. Note that

(1) $N \times_V f^*(\tilde{\mathbb{T}}^{n-r,r}) = h^*(f^*(\tilde{\mathbb{T}}^{n-r,r})) = \varprojlim h_i^*(f^*(\tilde{\mathbb{T}}^{n-r,r}))$ and;
(2) the completion of a direct limit of perfectoid affinoid (K, K°)-algebra is again perfectoid affinoid.

Therefore $(V, N \times_V f^*(\tilde{\mathbb{T}}^{n-r,r}))$ is affinoid perfectoid which covers (V, N). □

Lemma 4.8. *Assume that $(V, N) \in X_{\text{prolog}}$ is affinoid perfectoid with $\hat{N} = \operatorname{Spa}(R, R^+)$.*

(1) *For any nonzero element $b \in K^\circ$, we have $\mathcal{O}^+_{X_{\log}}((V, N))/b = R^+/b$ and it is almost equal to $(\mathcal{O}^+_{X_{\log}}/b)((V, N))$.*
(2) *The image of $(\mathcal{O}^+_{X_{\log}}/b_1)((V, N))$ in $(\mathcal{O}^+_{X_{\log}}/b_2)((V, N))$ is equal to R^+/b_2 for any nonzero nilpotent elements $b_1, b_2 \in K^\circ$ with $|b_1| < |b_2|$*
(3) *We have $\hat{\mathcal{O}}^+_{X_{\log}}((V, N)) = R^+$ and $\hat{\mathcal{O}}_{X_{\log}}((V, N)) = R$.*
(4) *The ring $\hat{\mathcal{O}}^+_{X_{\log}}((V, N))$ is the p-adic completion of $\mathcal{O}^+_{X_{\log}}((V, N))$.*
(5) *The cohomology groups $H^i((V, N), \hat{\mathcal{O}}^+_X)$ are almost zero for $i > 0$.*

Proof. The proof is almost identical to the proof of [Sch13a, Lemma 4.10]. We sketch the proof for the sake of completeness. As in the proof of [Sch13a, Lemma 4.10], it suffices to show that $N \to \mathcal{F}(N) = (\mathcal{O}^+_{\hat{N}}(\hat{N})/b)^a = (\mathcal{O}^+_N(N)/b)^a$ is a sheaf of almost K°-algebra, with $H^i(N, \mathcal{F}) = 0$ for $i > 0$.

Let N be a quasi-compact object being covered by $N_k \to N$. By quasi-compactness of N, we can assume that the covering consists of only one pro-log-étale morphism $N' \to N$. Write $N' = \varprojlim N'_i \to N'_0 \to N$, where $N'_0 \to N$ is log-étale morphism and $N'_i \to N'_j$ is surjective finite log-étale for $i > j \geq 0$. Note that the morphism q of a morphism $(W, M) \xrightarrow{(p,q)} (V, M')$ of $X_{\log}$ can be written as a composition of an étale morphism and a morphism of $W_{\text{f,log}}$, e.g. $M \to p^*M' \to M'$. Therefore, we can assume that $N'_0 \to N$ is induced by an étale morphism $V'_0 \to V$ of $X_{\text{ét}}$. Furthermore, by Lemma 4.6, the morphisms $N'_i \to N'_j$ are induced by finite étale morphisms of $(\pi(N'_j))_{\text{f,log}}$.

On the other hand, we have to show that the complex

$$\mathcal{C}(N', N) : 0 \to \mathcal{F}(N) \to \mathcal{F}(N') \to \mathcal{F}(N' \times_N N') \to \dots$$

is exact. Note that $\mathcal{F}(N') = \varinjlim \mathcal{F}(N'_j)$. So we have

$$\mathcal{C}(N', N) = \varinjlim \mathcal{C}(N'_i, N).$$

and one reduces to the case that $N' \to N$ is a composite of rational embeddings and finite étale maps. In this case, both N and N' are affinoid perfectoid, giving rise to perfectoid spaces $\hat{N}'$ and $\hat{N}$, and an étale cover $\hat{N}' \to \hat{N}$. Then Lemma 4.5 implies that

$$\mathcal{C}(N', N) : 0 \to (\mathcal{O}^+_{\hat{N}_{\text{ét}}}(\hat{N})/b)^a \to (\mathcal{O}^+_{\hat{N}_{\text{ét}}}(\hat{N}')/b)^a \to (\mathcal{O}^+_{\hat{N}_{\text{ét}}}(\hat{N}' \times_{\hat{N}} \hat{N}')/b)^a \to \dots$$

is exact. Note that $\mathcal{F}(N') = \varinjlim \mathcal{F}(N'_j)$. Therefore the statement follows from the vanishing of $H^i(W_{\text{ét}}, \mathcal{O}^{+a}_{W_{\text{ét}}}) = 0$ for $i > 0$ and any affinoid perfectoid space W, see [Sch12, Proposition 7.13]. □

Lemma 4.9. *Assume that (V, N) is an affinoid perfectoid, with $\hat{N} = \mathrm{Spa}(R, R^+)$. Let $\mathbb{L}$ be an $\mathbb{F}_p$-local system on $U = X \setminus D$. Then for all $i > 0$, the cohomology group*

$$H^i\bigg((V, N), \nu^*(u_X(\mathbb{L}) \otimes \mathcal{O}^+_{X_{\log}}/p)\bigg)^a = 0,$$

and it is almost finitely generated projective R^{+a}/p-module $M(N)$ for $i = 0$. If (V', N') is affinoid perfectoid, corresponding to $\hat{N}' = \mathrm{Spa}(R', R^{'+})$, and $(V', N') \to (V, N)$ some map in X_{prolog}, then $M(N') = M(N) \otimes_{R^{+a}/p} R'^{+a}/p$.

Proof. We just need to notice that $\nu^*\bigg(u_{X,*}(\mathbb{L})\bigg)$ will be extended to an $\mathbb{F}_p$-local system on N_k for some k in the index category of N (by Theorem 2.2). Therefore it follows from [Sch13a, Lemma 4.12]. □

5. Primitive Comparison

Following [Sch13a, Section 5], in this section we show the primitive comparison in our setting.

Theorem 5.1. *Let K be an algebraically closed complete extension of $\mathbb{Q}_p$, and let X be a proper smooth rigid analytic space over $\mathrm{Sp}(K)$ with an SSNC divisor D. Let $\mathbb{L}$ be an $\mathbb{F}_p$-local system on $(X - D)_{\text{ét}}$. Then there is an isomorphism of almost finitely generated $K^{\circ a}$-modules*

$$H^i(X_{\log}, u_{X,*}(\mathbb{L})) \otimes_{\mathbb{F}_p} K^{oa}/p \cong H^i(X_{\log}, u_{X*}(\mathbb{L}) \otimes \mathcal{O}^+_{X_{\log}}/p)$$

for all $i \geq 0$, where u_X is defined in Definition 2.7.

One can see the finiteness from the proof. We remark a more direct proof which is based on the primitive comparison of Scholze and functorial embedded resolution for rigid spaces over characteristic 0 fields due to Temkin.

Remark 5.2.

(1) Assume $\mathbb{L}$ comes from an $\mathbb{F}_p$-local system on X. Then by [Sch13a, Theorem 5.1], Theorem 2.3 and Theorem 2.8 we have that $H^i(X_{\log}, u_{X,*}(\mathbb{L}))$ is a finite dimensional $\mathbb{F}_p$ vector space for all $i \geq 0$, which vanishes for $i > 2 \dim X$.

(2) In general, by [Han17, Theorem 1.6] and [Tem17, Theorem 1.1.13], we can find a U' finite étale over U such that
- $\mathbb{L}|_{U'} \cong \mathbb{F}_p^{\oplus r}$ and;
- U' admits a smooth compactification with complement divisor SSNC.

Hence we have the finiteness for $H^i(X_{\log}, u_{X,*}(\mathbb{L}))$ as in (1).

Lemma 5.3. *Let k be a complete nonarchimedean field. Let V be an affinoid smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Let $D = \bigcup_{i=1}^l D_i$ be an SSNC divisor in V and let $x \in D_{i_1 i_2 \cdots i_r} \setminus \bigcup_{j \in \{1,2,\dots,l\} \setminus \{i_1, i_2, \dots, i_r\}} D_{i_1 i_2 \cdots i_r j}$ with closure $M = \overline{\{x\}}$. Then there* *exists a rational subset $U \subset V$ containing M, with $U \cong S \times \mathbb{D}^m(\underline{s})$, together with an étale map $S \xrightarrow{\phi} \mathbb{T}^{n-r}$ satisfying the following two conditions:*

1. *ϕ factors as a composite of rational embeddings and finite étale maps and;*
2. *$D_{i_j} \cap U$ is given by the vanishing locus of s_j and if $i \notin \{i_1, i_2, \dots, i_r\}$ then $D_i \cap U = \varnothing$.*

Proof. By [Mit09, Theorem 2.11] we may first find a rational subset $U_0 \subset V$ containing M such that $U_0 \cong S_0 \times \mathbb{D}^r(\underline{s})$, where S_0 is a smooth affinoid, satisfying our condition (2). Note that one can find such a rational containing M because of condition (1) of [Mit09, Theorem 2.11].

Now we may apply [Sch13a, Lemma 5.2] to our $(x, S_0 \times \{0\})$ to find a rational subset $S \subset S_0$ together with ϕ satisfying our condition (1). □

Lemma 5.4. *Let k be a complete nonarchimedean field. Let X be a proper smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Let $\bigcup_{i \in I} D_i = D \subset X$ be an SSNC divisor where $I = \{1, 2, \dots, r\}$. For any integer $N \geq 1$ and N distinct elements $\gamma_N < \gamma_{N-1} < \cdots < \gamma_1 = 1$ in the norm group Γ of k, one may find N finite covers $\mathcal{U}^{(i)} = \bigcup_{J \subset I} \{U_l^{J,(i)}\}$ of X by affinoid open subsets. Here $U_l^{J,(i)} \cong S_l^{J,(i)} \times \mathbb{D}^{|J|}(\underline{s}/p^{\gamma_i})$ where $S_l^{J,(i)}$ (viewed as $S_l^{J,(i)} \times \{0\}$) are affinoid open subsets of D_J, such that the following conditions hold:*

1. *$D \cap U_l^{J,(i)}$ is given by vanishing locus of coordinates on the disc;*
2. *For all i, J and l, the closure $\overline{S_l^{J,(i+1)}}$ of $S_l^{J,(i+1)}$ in D_J is contained in $S_l^{J,(i)}$. Hence the closure of $U_l^{J,(i+1)}$ in X is contained in $U_l^{J,(i)}$;*
3. *For all l and J, $S_l^{J,(N)} \subset \ldots \subset S_l^{J,(1)}$ is a chain of rational subsets. Hence the same holds for $U_l^{J,(i)}$'s;*
4. *For J, J', l and l', the intersection $U_l^{J,(1)} \cap U_{l'}^{J',(1)} \subset U_l^{J,(1)}$ is a rational subset and;*
5. *For all l and J, there is an étale map $S_l^{J,(1)} \to \mathbb{T}^{n-|J|}$ that factors as a composite of rational subsets and finite étale maps.*

Proof. The proof is almost identical to that of [Sch13a, Lemma 5.3] except we use Lemma 5.3 to replace [Sch13a, Lemma 5.2] in the argument. □

Lemma 5.5. *Let K be a complete non-archimedean field extension of $\mathbb{Q}_p$ that contains all roots of unity; choose a compatible system $\zeta_l \in K$ of l-th roots of unity. Let*

$$R_0 = \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_{n-r}^{\pm 1}, T_{n-r+1}, \dots, T_n\rangle,$$
$$R' = \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_{n-r}^{\pm 1/p^\infty}, T_{n-r+1}^{1/\infty}, \dots, T_n^{1/\infty}\rangle$$

where $T^{1/\infty}$ means adjoining all power roots of T, and

$$R = \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_{n-r}^{\pm 1/p^\infty}, T_{n-r+1}^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle.$$

Let S_0 be an R_0-algebra which is p-adically complete flat over $\mathbb{Z}_p$ with the p-adic topology. Let $\Delta \coloneqq \mathbb{Z}_p^{n-r} \times \widehat{\mathbb{Z}}^r$ such that the k-th basis vector acts on R' via

$$\prod T_j^{i_j} \mapsto \zeta^{i_k} \prod T_j^{i_j},$$

where $\zeta^{i_k} = \zeta_l^{i_k l}$ whenever $i_k l \in \mathbb{Z}$. Let $\Delta \twoheadrightarrow \Delta_\infty \coloneqq \mathbb{Z}_p^n$ be the obvious projection. Then

(1) $H^q_{cont}(\Delta_\infty, S_0/p^m \otimes_{R_0/p^m} R/p^m) \to H^q_{cont}(\Delta, S_0/p^m \otimes_{R_0/p^m} R'/p^m)$ *is an almost isomorphism,*
(2) $H^q_{cont}(\Delta_\infty, R/p^m)$ *is an almost finitely presented* R_0*-module for all* m*,*
(3) *the map*

$$\bigwedge^q R_0^n = H^q_{cont}(\Delta_\infty, R_0) \to H^q_{cont}(\Delta, R')(=^a H^q_{cont}(\Delta_\infty, R) \text{ by (1) above })$$

is injective with cokernel killed by $\zeta_p - 1$*,*
(4) $H^q_{cont}(\Delta_\infty, S_0/p^m \otimes_{R_0/p^m} R/p^m) = S_0/p^m \otimes_{R_0/p^m} H^q_{cont}(\Delta_\infty, R/p^m)$ *for all* m *and,*
(5) $H^q_{cont}(\Delta_\infty, S_0 \hat{\otimes}_{R_0} R) = S_0 \hat{\otimes}_{R_0} H^q_{cont}(\Delta_\infty, R)$

Proof. (1) follows from [Ols09, Lemma 3.10], notice that the action of Δ is continuous with respect to the p-adic topology on these Δ-modules. (2) to (5) follows from (the proof of) [Sch13a, Lemma 5.5]. □

Lemma 5.6. *Let* K *be as in the previous Lemma. Let* (V, N) *be an object in* $X_{\log}$ *with an étale map* $V \to \mathbb{T}^{n-r,r}$ *as one of the* $V_k^{J,(1)}$*'s in Lemma 5.4. Let* $\mathbb{L}$ *be an* $\mathbb{F}_p$*-local system on* $U_{\text{ét}}$*. Then*

(1) *For* $i > n = \dim X$*, the cohomology group*

$$H^i((V, N), (u_{X,*}\mathbb{L}) \otimes \mathcal{O}^+_{X_{\log}}/p)$$

is almost zero as $\mathcal{O}_K$*-module.*
(2) *Assume* $V' \subset V$ *is a rational subset which is strictly contained in* V*. Then the image of*

$$H^i((V, N), (u_{X,*}\mathbb{L}) \otimes \mathcal{O}^+_{X_{\log}}/p) \to H^i((V', N' = V' \times_V N), (u_{X,*}\mathbb{L}) \otimes \mathcal{O}^+_{X_{\log}}/p)$$

is an almost finitely generated $\mathcal{O}_K$*-module.*

Proof. This follows from the proof of [Sch13a, Lemma 5.6]. In the argument we need to replace [Sch13a, Lemma 4.5] by Lemma 4.5 and Lemma 4.6, [Sch13a, Lemma 4.12] by Lemma 4.9, [Sch13a, Lemma 5.3] by Lemma 5.4 and [Sch13a, Lemma 5.5] by Lemma 5.5. □

Lemma 5.7. *Let* K *be a perfectoid field of characteristic* 0 *containing all* p*-power roots of unity. Let* $\mathbb{L}$ *be an* $\mathbb{F}_p$*-local system on* $U_{\text{ét}}$*. Then*

$$H^j(X_{\log}, (u_{X*}\mathbb{L}) \otimes \mathcal{O}^+_X/p)$$

is an almost finitely generated $\mathcal{O}_K$*-module, which is almost zero for* $j > 2\dim X$*.*

Proof. Consider the projection $\mu \colon X_{\log} \to X_{an}$ sending U to (U, U). Previous Lemma 5.6 shows that $R^j\lambda_*(u_{X*}\mathbb{L} \otimes \mathcal{O}^+_X/p)$ is almost zero for $j > \dim X$. Notice that any covering of (X, X) in $X_{\log}$ can be refined by ones meeting the condition of previous Lemma. The cohomological dimension of X_{an} is $\leq \dim X$ by [dJvdP96, Proposition 2.5.8], we get the desired vanishing result. The proof above is similar to the counterpart of [Sch13a, Lemma 5.8].

The proof of almost finitely generatedness is also similar to that in [Sch13a, Lemma 5.8]. Again, we have to replace [Sch13a, Lemma 5.3] by Lemma 5.4 and [Sch13a, Lemma 5.6] by Lemma 5.6. □

Definition 5.8. Let (X, D) be as before. The tilted integral structure sheaf $\hat{\mathcal{O}}^+_{X^\flat_{\log}}$ is given by $\varprojlim \mathcal{O}^+_{X_{\log}}/p$ where the inverse limit is taken along the Frobenius map. Set $\hat{\mathcal{O}}_{X^\flat_{\log}} = \hat{\mathcal{O}}^+_{X^\flat_{\log}}[\frac{1}{p}]$.

The next lemma follows from repeating the argument of its untilted version (Lemma 4.8).

Lemma 5.9. *Let K be a perfectoid field of characteristic 0, and let X be an adic space associated to a rigid space over $\mathrm{Sp}(K)$. Let $N \in X_{\mathrm{prolog}}$ be affinoid perfectoid, with $\hat{N} = \mathrm{Spa}(R, R^+)$ where (R, R^+) is a perfectoid affinoid (K, K^o)-algebra. Let $(R^\flat, R^{\flat+})$ be its tilt. Then we have*

(1) $\hat{\mathcal{O}}^+_{X^\flat_{\log}}(N) = R^{\flat+}$ *and* $\hat{\mathcal{O}}_{X^\flat_{\log}}(N) = R^\flat$*;*
(2) *The cohomology groups $H^i(N, \hat{\mathcal{O}}^+_{X^\flat_{\log}})$ are almost zero for $i > 0$, with respect to the almost setting defined by the maxiaml ideal of topologically nilpotent elements in K°.*

Now we can follow Scholze's method to show Theorem 5.1.

Proof of Theorem 5.1. To simplify our notations, throughout the proof, we still denote $\nu^*(u_{X,*}(\mathbb{L}))$ by $u_{X,*}(\mathbb{L})$. Note that $K^\flat$ is an algebraically closed field of characteristic p. Fix an element $\pi \in \mathcal{O}_{K^\flat}$ such that $(\pi)^\sharp = p$. Note that $\hat{\mathcal{O}}^+_{X^\flat_{\log}}$ is a sheaf of perfect flat $\mathcal{O}_{K^\flat}$-algebras with $\hat{\mathcal{O}}^+_{X^\flat_{\log}}/\pi^k = \mathcal{O}^+_X/p^k$ (by Lemma 5.9 and Lemma 4.8). Let $M_k = H^i(X_{\mathrm{prolog}}, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}^+_{X^\flat_{\log}}/\pi^k)^a$. It follows from Lemma 5.7 that M_k satisfy the hypotheses of [Sch13a, Lemma 2.12]. Hence there is some $r \in \mathbb{N}$ such that $M_k = (K^{\flat o}/\pi^k)^r$ as almost $K^\flat$-modules, compatibly with the Frobenius action. By Theorem 4.7, Lemma 5.9 and [Sch13a, Lemma 3.18], we have

$$R\varprojlim(u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}^+_{X^\flat_{\log}}/\pi^k)^a = (u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}^+_{X^\flat_{\log}})^a.$$

Therefore, we have

$$H^i(X_{\mathrm{prolog}}, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}^+_{X^\flat_{\log}})^a \cong \varprojlim H^i(X_{\mathrm{prolog}}, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}^+_{X^\flat_{\log}}/\pi^k)^a \cong (\mathcal{O}^a_{K^\flat})^r.$$

Note that the site X_{prolog} is algebraic and the final object $(X, X) \in X_{\mathrm{prolog}}$ is coherent. We invert π and get

$$H^i(X_{\mathrm{prolog}}, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}_{X^\flat_{\log}}) \cong (K^\flat)^r$$

which is still compatible with the action of Frob. Then we use the Artin-Schreier sequence

$$0 \to u_{X,*}(\mathbb{L}) \to u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}_{X^\flat_{\log}} \xrightarrow{h} u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}_{X^\flat_{\log}} \to 0$$

where the map h sends $v \otimes f$ to $v \otimes (f^p - f)$. This is an exact sequence of sheaves: by Lemma 2.9, $u_{X,*}(\mathbb{L})$ is locally coming from a $\mathbb{F}_p$-local system on $X_{\text{ét}}$, moreover, $u_{X,*}(\mathbb{F}_p) = \mathbb{F}_p$ on $X_{\log}$. Therefore, it suffices to check the map h is surjective locally on affinoid perfectoid $N \in X_{\mathrm{prolog}}$ and over which $u_{X,*}(\mathbb{L})$ is trivial. Note that $\hat{N}^\flat_{\text{Fét}} \cong \hat{N}_{\text{Fét}}$, and finite étale covers of $\hat{N}$ come via pullback from finite étale covers in X_{prolog} by [Sch12, Lemma 7.5 (i)].

Denote X_{prolog} by X. The Artin-Schreier sequence gives

$$\begin{array}{ccccc} \dots H^i(X, u_{X,*}(\mathbb{L})) & \longrightarrow & H^i(X, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}_{X^\flat_{\log}}) & \longrightarrow & H^i(X, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}_{X^\flat_{\log}}) \dots \\ \| & & \| & & \| \\ \mathbb{F}_p^r & \longrightarrow & (K^\flat)^r & \longrightarrow & (K^\flat)^r \end{array}$$

where the map $(K^\flat)^r \to (K^\flat)^r$ is coordinate-wise $x \mapsto x^p - x$. The map $(K^\flat)^r \to (K^\flat)^r$ is surjective since $K^\flat$ is algebraically closed. Using Lemma 3.9 (2), we have

$$H^i(X_{\log}, u_{X,*}(\mathbb{L})) = H^i(X_{\text{prolog}}, u_{X,*}(\mathbb{L})) = H^i(X_{\text{prolog}}, u_{X,*}(\mathbb{L}) \otimes \hat{\mathcal{O}}_{X^\flat_{\log}})^{\text{Frob}=id} = \mathbb{F}_p^r.$$

which implies the theorem. □

Remark 5.10. By the same proof, one has the following variant of Theorem 5.1: let X be a proper smooth rigid analytic space over $\text{Sp}(k)$ with an SSNC divisor D. Let $\mathbb{L}$ be an $\mathbb{F}_p$-local system on $(X - D)_{\text{ét}}$. Then there is an isomorphism of almost finitely generated $\hat{\bar{k}}^{\circ a}$-modules

$$H^i((X, X_{\bar{k}}), \nu^*(u_{X,*}(\mathbb{L}))) \otimes_{\mathbb{F}_p} \hat{\bar{k}}^{\circ a}/p \cong H^i((X, X_{\bar{k}}), \nu^*(u_{X*}(\mathbb{L})) \otimes \mathcal{O}^+_{X_{\log}}/p)$$

for all $i \geq 0$. Here $X_{\bar{k}}$ is the pro-system of X_l where l/k runs through all finite extension of k, see also [Sch13a, Proposition 3.15] and the discussion before it.

6. The period sheaves

Definition 6.1. On $X_{\log}$ we have the *sheaf of log differentials*

$$\Omega^1_{X_{\log}}(\log D) := \lambda^{-1}(\Omega^1_X(\log D)) \bigotimes_{\lambda^{-1}(\mathcal{O}_X)} \mathcal{O}_{X_{\log}}$$

where $\lambda \colon X_{\log} \to X_{\text{ét}}$ is the natural map sending $(V \to X)$ to (V, V). Note that this is a locally finite free sheaf of $\mathcal{O}_{X_{\log}}$-modules.

The following definitions are similar to [Sch13a, Definition 6.1].

Definition 6.2. Let X be a rigid space over $\text{Sp}(k)$ with SSNC divisor D. We have the following sheaves on X_{prolog}.

(1) The sheaf $\mathbb{A}_{\inf} := W(\hat{\mathcal{O}}^+_{X^\flat_{\log}})$ and its rational version $\mathbb{B}_{\inf} := \mathbb{A}_{\inf}[\frac{1}{p}]$. We have $\theta : \mathbb{A}_{\inf} \to \hat{\mathcal{O}}^+_{X_{\log}}$ extended to $\theta : \mathbb{B}_{\inf} \to \hat{\mathcal{O}}_{X_{\log}}$.
(2) The positive de Rham sheaf is given by $\mathbb{B}^+_{\text{dR}} := \varprojlim \mathbb{B}_{\inf}/(\ker\theta)^n$ with its filtration $\text{Fil}^i\mathbb{B}^+_{\text{dR}} = (\ker\theta)^i\mathbb{B}^+_{\text{dR}}$.
(3) The de Rham sheaf $\mathbb{B}_{\text{dR}} = \mathbb{B}^+_{\text{dR}}[t^{-1}]$, where t is any element that generates $\text{Fil}^1\mathbb{B}^+_{\text{dR}}$. It has the filtration $\text{Fil}^i\mathbb{B}_{\text{dR}} = \sum_{j\in\mathbb{Z}} t^{-j}\text{Fil}^{i+j}\mathbb{B}^+_{\text{dR}}$.

The analogue of [Sch13a, 6.2-6.7] holds in our setting with the same proof, let us summarize it in the following:

Remark 6.3. Let K be a perfectoid field which is the completion of some algebraic extension of k and fix $\pi \in K^\flat$ such that $\pi^\sharp/p \in (K^\circ)^\times$. Let (V, N) be an affinoid perfectoid in the localized site $X_{\text{prolog}}/\text{Spa}(K, K^\circ)$ with $\hat{N} = \text{Spa}(R, R^+)$. Then we have

(1) There is an element $\xi \in \mathbb{A}_{\mathrm{inf}}(K, K^\circ)$ that generates $\ker(\theta\colon \mathbb{A}_{\mathrm{inf}}(R, R^+) \to R^+)$, and is not a zero-divisor in $\mathbb{A}_{\mathrm{inf}}(R, R^+)$.
(2) we have a canonical isomorphism

$$\mathbb{A}_{\mathrm{inf}}(V, N) = \mathbb{A}_{\mathrm{inf}}(R, R^+),$$

and analogous statements hold for $\mathbb{B}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathbb{B}_{\mathrm{dR}}$. In particular, $\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}^+(V, N)$ is a principal ideal in $\mathbb{B}_{\mathrm{dR}}^+$ generated by a non-zero-divisor $\xi \in \mathbb{A}_{\mathrm{inf}}(K, K^\circ)$.
(3) All $H^i((V, N), \mathcal{F})$ are almost zero for $i > 0$, where $\mathcal{F}$ is any of the sheaves above. In particular,

$$\mathrm{gr}^\bullet\mathbb{B}_{\mathrm{dR}}(V, N) = \mathrm{gr}^\bullet\mathbb{B}_{\mathrm{dR}}(R, R^+) = R[\xi^{\pm 1}].$$

(4) Let S be a profinite set, and let $(V, N') = (V, N \times S) \in X_{\mathrm{prolog}}$ which is again affinoid perfectoid. Then

$$\mathcal{F}(V, N') = \mathrm{Hom}_{cont}(S, \mathcal{F}(V, N))$$

for any of the sheaves

$$\mathcal{F} \in \{\hat{\mathcal{O}}_{X_{\log}}, \hat{\mathcal{O}}^+_{X_{\log}}, \hat{\mathcal{O}}_{X^\flat_{\log}}, \hat{\mathcal{O}}^+_{X^\flat_{\log}}, \mathbb{A}_{\mathrm{inf}}, \mathbb{B}_{\mathrm{inf}}, \mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}, \mathrm{gr}^i\mathbb{B}_{\mathrm{dR}}\}.$$

For all $i \in \mathbb{Z}$, we have $\mathrm{gr}^i\mathbb{B}_{\mathrm{dR}} \cong \hat{\mathcal{O}}_{X_{\log}}(i)$ as sheaves on X_{prolog} where (i) denotes a Tate twist in the same sense as in [Sch13a, Proposition 6.7].

Definition 6.4. On $X_{\log}$ we have the *sheaf of log differentials*

$$\Omega^1_{X_{\log}}(\log D) := \lambda^{-1}(\Omega^1_X(\log D)) \bigotimes_{\lambda^{-1}(\mathcal{O}_X)} \mathcal{O}_{X_{\log}}$$

where $\lambda\colon X_{\log} \to X_{\text{ét}}$ is the natural morphism of sites sending $(V \to X)$ to (V, V). Note that this is a locally finite free sheaf of $\mathcal{O}_{X_{\log}}$-modules.

Remark 6.5. Note that the (V, N)'s in $X_{\log}$ satisfying the following conditions:
(1) V (hence N) is an affinoid space;
(2) there is an étale morphism $V \to \mathbb{T}^{n-r,r}(\underline{Z})$ such that

$$g^{-1}(D) = \bigcup_{l=n-r+1}^{n} V(Z_l)$$

where $g\colon V \to X$ is the structure map;
(3) there is a finite étale morphism $N \to V[\sqrt[m]{Z_l}]$;

form a basis of $X_{\log}$ by Theorem 2.2 and Lemma 5.3. For (V, N) satisfying the above conditions with $N = \mathrm{Sp}(R)$, we have an isomorphism

$$\Omega^1_{X_{\log}}(\log D)(V, N) \cong \bigoplus_{1 \le l \le n} R \cdot \frac{dZ_l}{Z_l}.$$

Hence for such a (V, N), we have $\Omega^1_{X_{\log}}(\log D)(V, N) = \Omega^1_N(\log(f^{-1}D))(N)$. Here $f\colon N \to X$ is induced from $N \to V \to X$.

Definition 6.6. Let X be a smooth rigid adic space over $\mathrm{Sp}(k)$ where k is a discretely valued complete non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field κ. Consider the following sheaves on X_{prolog}.

(1) The sheaf of differentials

$$\Omega^1_X(\log D) := \nu^*\bigg(\Omega^1_{X_{\log}}(\log D)\bigg).[2]$$

We also define $\Omega^i_X(\log D) := \wedge^i \Omega^1_X(\log D)$.

(2) The positive logarithmic structural de Rham sheaf $\mathcal{O}\mathbb{B}^+_{\log \mathrm{dR}}$ is given by the sheafification of the presheaf sending affinoid perfectoid (V, N) with

$$N = \varprojlim N_i = \varprojlim \mathrm{Sp}(R_i) \text{ and } \hat{N} = \mathrm{Sp}a(R, R^+)$$

to the colimit over i of

$$(\boxdot) \qquad \varprojlim_r \frac{\bigg(\big(R_i^\circ \hat{\otimes}_{W(\kappa)} (\mathbb{A}_{\inf}(R, R^+)/\ker(\theta)^r)\big)[\frac{1\otimes[f_k^\flat]}{f_k\otimes 1}][\frac{1}{p}]\bigg)}{\ker(1\otimes\theta)^r}$$

Here $\{f_k \in \mathcal{O}(V)\}$ are defining functions of D_k given as part of the definition of (V, N) being affinoid perfectoid. The completed tensor product is the p-adic completion of the tensor product. Here $1 \otimes \theta$ is the tensor product of the map $R_i^\circ \to R^+$ and $\theta\colon \mathbb{A}_{\inf}(R, R^+) \to R^+$, moreover it sends $\frac{1\otimes[f_k^\flat]}{f_k\otimes 1}$ to 1. Note that R contains all roots of f_k, therefore we have $f_k^\flat = (f_k, (f_k)^{\frac{1}{p}}, \ldots,) \in (R^+)^\flat$, in particular $\theta([f_k^\flat]) = f_k$.

(3) The uncompleted logarithmic structure de Rham sheaf is given by $\mathcal{O}\mathbb{B}^{uc}_{\log \mathrm{dR}} := \mathcal{O}\mathbb{B}^+_{\log \mathrm{dR}}[t^{-1}]$ where t is a generator of $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}$.

It is clear that we still have the map $\theta : \mathcal{O}\mathbb{B}^+_{\log \mathrm{dR}} \to \hat{\mathcal{O}}^+_X$ which induces its filtration

$$\mathrm{Fil}^i\mathcal{O}\mathbb{B}^+_{\log \mathrm{dR}} = (\ker\theta)^i\mathcal{O}\mathbb{B}^+_{\log \mathrm{dR}}.$$

We also have a filtration on $\mathcal{O}\mathbb{B}^{uc}_{\log \mathrm{dR}}$ by

$$\mathrm{Fil}^i\mathcal{O}\mathbb{B}^{uc}_{\log \mathrm{dR}} = \sum_{j\in\mathbb{Z}} t^{-j}\mathrm{Fil}^{i+j}\mathcal{O}\mathbb{B}^+_{\log \mathrm{dR}}.$$

(4) Finally, the logarithmic structure de Rham sheaf is defined to be the completion of uncompleted logarithmic structure de Rham sheaf with respect to the filtration defined above[3]

$$\mathcal{O}\mathbb{B}_{\log \mathrm{dR}} := \widehat{\mathcal{O}\mathbb{B}^{uc}_{\log \mathrm{dR}}}.$$

Note that $\mathcal{O}\mathbb{B}_{\log \mathrm{dR}}$ is equipped with the filtration coming from that on $\mathcal{O}\mathbb{B}^{uc}_{\log \mathrm{dR}}$, with respect to which it is complete, and that both two sheaves have the same graded pieces.

Remark 6.7. (1) It is easy to check that the colimit over i of $\boxdot$ does not depend on the presentation of N, and it does define a presheaf.

(2) Later on we will see that for a set of basis (V, N) of X_{prolog}, there is a cofinal system of i's such that the outcomes $\boxdot$ corresponding to i are the same, see Proposition 6.8.

[2]One should notice the difference between ν^{-1} and ν^*.

[3]We thank Xinwen Zhu for pointing out to us that the original sheaf we defined was not complete, and we need to take completion with respect to this filtration, c.f. [DLLZ18, Remark 3.11].

(3) Note that we have a natural $\mathbb{B}_{\mathrm{dR}}^{+}$-linear connection with log poles:

$$\mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{+} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{+} \otimes_{\mathcal{O}_{X_{\log}}} \Omega^1_{X_{\log}}(\log D)$$

sending

$$\frac{1\otimes [f_k^\flat]}{f_k\otimes 1} \mapsto -\frac{1\otimes [f_k^\flat]}{(f_k\otimes 1)^2}\, df_k = -\frac{1\otimes [f_k^\flat]}{f_k\otimes 1}\cdot d\log(f_k),$$

extended from the connection $\mathcal{O}_{X_{\log}} \xrightarrow{\nabla} \Omega^1_{X_{\log}}(\log D)$. Because $t \in \mathbb{B}_{\mathrm{dR}}^{+}$, inverting it, we get a natural $\mathbb{B}_{\mathrm{dR}}$-linear connection with log poles:

$$\mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{uc} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{uc} \otimes_{\mathcal{O}_{X_{\log}}} \Omega^1_{X_{\log}}(\log D).$$

Take completion with respect to the induced filtration, we get:

$$\mathcal{O}\mathbb{B}_{\log \mathrm{dR}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\log \mathrm{dR}} \otimes_{\mathcal{O}_{X_{\log}}} \Omega^1_{X_{\log}}(\log D).$$

(4) The definition of these de Rham period sheaves uses the fact that X is defined over a p-adic field. This is the crucial place where we have to use this fact.[4]

We describe $\mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{+}$ in the following proposition (see also [Sch13a, Proposition 6.10] and [Sch16]). Let $U \subset X$ be an open. Let K be a perfectoid field which is the completion of an algebraic extension of k. We get the base change U_K of U to $\mathrm{Sp}(K)$, and again consider $U_K \in X_{\mathrm{prolog}}$ by slight abuse of notation. Let $\varphi : U \to \mathbb{T}^{n-r,r}(\underline{Z})$ (cf. Example 4.4) be an étale morphism such that $f_k := \varphi^*(Z_{n-r+k})$ $(k = 1, \dots, r)$ defines the component D_k of $D \cap U$. Note that such U's form a basis of X. Let $\tilde{U} = U \times_{\mathbb{T}^{n-r,r}} \tilde{\mathbb{T}}^{n-r,r}$. Taking a further base change to K, we get $(U_K, \tilde{U}_K) \in X_{\mathrm{prolog}}$ is perfectoid.

Proposition 6.8. *Let notations be as above. Consider the localized site $X_{\mathrm{prolog}}/(U_K, \tilde{U}_K)$. We have the elements*

$$u_i = Z_i \otimes 1 - 1 \otimes [Z_i^\flat] \in \mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{+}|_{(U,\tilde{U})}$$

for $i = 1, \dots, n-r$, and

$$u_j = 1 - \frac{1\otimes [Z_j^\flat]}{Z_j \otimes 1} \in \mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{+}|_{(U,\tilde{U})}$$

for $j = n-r+1, \dots, n$. Here we abuse the notations by using Z_j to denote $\varphi^*(Z_j) = f_j$. We will also use Z_j (resp. $[Z_j^\flat]$) to denote $Z_j \otimes 1$ (resp. $[Z_j^\flat]\otimes 1$) to simplify our notations.

The map

$$\mathbb{B}_{\mathrm{dR}}^{+}|_{(U_K,\tilde{U}_K)}[\![X_1, \dots, X_n]\!] \to \mathcal{O}\mathbb{B}_{\log \mathrm{dR}}^{+}|_{(U_K,\tilde{U}_K)}$$

sending X_i to u_i is an isomorphism of sheaves over $X_{\mathrm{prolog}}/(U_K, \tilde{U}_K)$.

Proof. **Step 0**: definition of the map.

Let (V, N) be an affinoid perfectoid over $(U_K, \tilde{U}_K)$ where $N = \varprojlim N_i$ with $N_i = \mathrm{Spa}(R_i, R_i^\circ)$ and $\hat{N} = \mathrm{Spa}(R, R^+)$. For each r and i, we use the fact that

$$\frac{\mathbb{B}_{\mathrm{dR}}^{+}(R, R^+)[\![X_1, \dots, X_n]\!]}{(\xi, X_i)^r} \cong \frac{\mathbb{A}_{inf}(R, R^+)[\![X_1, \dots, X_n]\!][1/p]}{(\xi, X_i)^r}$$

[4] We thank Bhargav Bhatt for reminding us this in a private communication.

to define the morphism

$$\frac{\mathbb{A}_{inf}(R,R^+)[\![X_1,\ldots,X_n]\!][1/p]}{(\xi,X_i)^r} \to \frac{\Big(\big(R_i^\circ\hat{\otimes}_{W(\kappa)}(\mathbb{A}_{\inf}(R,R^+)/\ker(\theta)^r)\big)[\frac{1\otimes[f_k^\flat]}{f_k\otimes 1}][\frac{1}{p}]\Big)}{\ker(1\otimes\theta)^r} =: S_{i,r},$$

by sending any element $a \in \mathbb{A}_{inf}(R,R^+)$ to $1\otimes a$ and X_i to u_i as described in the statement of this proposition. Here we used the fact that the ideal (ξ, X_i) is sent inside $\ker(1\otimes\theta)$. Taking inverse limit over r and then colimit over i gives the morphism in the statement of this proposition.

We want to show that for any N_i there exists a higher $N_{i'} \to N_i$ such that the morphism

$$(\boxdot)\qquad \frac{\mathbb{A}_{inf}(R,R^+)[\![X_1,\ldots,X_n]\!][1/p]}{(\xi,X_i)^r} \to S_{i',r}$$

is an isomorphism for all r. This shows in particular that in Definition 6.6(2), there is a cofinal system of i's for which the outcomes ($\boxdot$) are the same.

Step 1: construct a section.

Let i be large enough, so that we get a log étale morphism $(V, N_i) \to \mathbb{T}^{n-r,r}$ where $N_i = \mathrm{Spa}(R_i, R_i^\circ)$. By Theorem 2.2, we see that there is an $m \in \mathbb{N}$ such that $(N_i \times_{\mathbb{T}^{n-r,r}} \mathbb{T}^{n-r,r}[\sqrt[m]{Z_l}])^\nu =: \mathrm{Spa}(R_{i'}, R_{i'}^\circ) \to \mathbb{T}^{n-r,r}[\sqrt[m]{Z_l}]$ is étale. We will take $\mathrm{Spa}(R_{i'}, R_{i'}^\circ)$ to be the $N_{i'}$ we want.

To simplify the notations further, let us denote $\mathbb{B}_r := \frac{\mathbb{B}^+_{\mathrm{dR}}(R,R^+)[\![X_1,\ldots,X_n]\!]}{(\xi,X_i)^r}$. For technical reason we also want to consider, for each r, the $\mathbb{B}^+_{\mathrm{dR}}(R,R^+)$-algebra $\mathbb{B}'_r := \frac{\mathbb{B}^+_{\mathrm{dR}}(R,R^+)[X_1,\ldots,X_{n-r},\tilde{X}_{n-r+1},\ldots,\tilde{X}_n]}{(\xi,X_1,\ldots,X_{n-r},\tilde{X}_{n-r+1},\ldots,\tilde{X}_n)^r}$. There is a natural morphism $\mathbb{B}'_r \xrightarrow{\beta_r} \mathbb{B}_r$ where $\beta_r(\tilde{X}_l) = \frac{[(Z_l^{1/m})^\flat]}{(1-X_l)^{1/m}} - [(Z_l^{1/m})^\flat]$. Note that $\frac{1}{(1-X_l)^{1/m}}$ can be written as a power series in $\mathbb{Q}[\![X_l]\!]$, hence our expression makes sense. We still denote the composition $\theta\circ\beta_r$ by θ.

In the following, we will show that there is a natural morphism $R_{i'} \to \mathbb{B}'_r$, whose image is contained in a open and bounded (w.r.t. the p-adic topology induced from $\mathbb{B}_r$) subring inside $\mathbb{B}'_r$, which is compatible with θ map for all r.

First note that for all r, there is a map

$$W(\kappa)[p^{-1}][Z_1^{\pm 1},\ldots,Z_{n-r}^{\pm 1}, Z_{n-r+1}^{1/m},\ldots,Z_n^{1/m}] \to \mathbb{B}'_r$$

by sending $Z_j \mapsto X_j + [Z_j^\flat]$ for $j \le n-r$ and $Z_l^{1/m} \mapsto \tilde{X}_l$ for all $l > n-r$.

Now we need the following lemma.

Lemma 6.9. *Let $\mathcal{O}$ be an excellent complete rank 1 valuation ring with a pseudo-uniformizer ϖ, and let F be its fraction field which is viewed as a non-archimedean field. Let A_0^+ be a finitely presented flat $\mathcal{O}$-algebra. Let $A = \widehat{A_0^+}[1/p]$, where the completion is with respect to ϖA_0^+, which is an affinoid F-algebra. Let $U = \mathrm{Sp}(B)$ be an affinoid rigid space admitting an étale map $U \to \mathrm{Sp}(A)$. Then there exists a finitely presented $\mathcal{O}$-flat A_0^+-algebra B_0^+, such that $B_0 = B_0^+[1/p]$ is étale over $A_0^+[1/p]$ and B° is the ϖ-adic completion of B_0^+.*

Proof. This is a slight generalization of [Sch13a, Lemma 6.12] and it follows from the same proof as [Sch13a, Proof of Lemma 6.12]. □

Apply the above lemma to $\mathcal{O} = W(\kappa)$, $A_0^+ = W(\kappa)[Z_1^{\pm 1}, \ldots, Z_{n-r}^{\pm 1}, Z_{n-r+1}^{1/m}, \ldots, Z_n^{1/m}]$ and $B = R_{i'}$ gives a finitely generated $W(\kappa)[Z_1^{\pm 1}, \ldots, Z_{n-r}^{\pm 1}, Z_{n-r+1}^{1/m}, \ldots, Z_n^{1/m}]$-algebra $R_{i'0}^{\circ}$ whose generic fibre $R_{i'0}$ is étale over

$$W(\kappa)[p^{-1}][Z_1^{\pm 1}, \ldots, Z_{n-r}^{\pm 1}, Z_{n-r+1}^{1/m}, \ldots, Z_n^{1/m}].$$

By Hensel's Lemma, we get a unique lift $R_{i'0} \to \mathbb{B}_r'$. In particular we get a lift of $R_{i'0}^{\circ}$. This extends to the p-adic completion with image lands in an open bounded subring (see [Sch13a, Lemma 6.11 and its proof]). Hence we get a lift of $R_{i'} \to \mathbb{B}_r'$ with image lands in an open bounded subring.

Step 2: injectivity of ⚁.

After composing with β_r, we get a map (recall that $\frac{[Z_l^\flat]}{Z_l} = 1 - X_l$)

$$S_{i',r} \to \mathbb{B}_r$$

for which the composition

$$\mathbb{B}_r \to S_{i',r} \to \mathbb{B}_r$$

is the identity. Therefore we see that ⚁ is injective.

Step 3: surjectivity of ⚁.

Now we only need to show that

$$\mathbb{B}_r \to S_{i',r}$$

is surjective. Let us consider the following commutative diagrams

$$\begin{array}{ccc}
\mathbb{B}_r' & \xrightarrow{\alpha_r} & \frac{(R_{i'}^{\circ}\hat{\otimes}_{W(\kappa)}(\mathbb{A}_{\inf}(R,R^+)/\ker(\theta)^r))[\frac{1}{p}]}{\ker(1\otimes\theta)^r} \\
\Big\downarrow{\scriptstyle \beta_r} & & \Big\downarrow{\scriptstyle \epsilon_r} \\
\mathbb{B}_r & \xrightarrow{\gamma_r} & S_{i',r}
\end{array}$$

and

$$\begin{array}{ccc}
\frac{(R_{i'}^{\circ}\hat{\otimes}_{W(\kappa)}(\mathbb{A}_{\inf}(R,R^+)/\ker(\theta)^r))[\frac{1}{p}]}{\ker(1\otimes\theta)^r} & \xrightarrow{\epsilon_r} & S_{i',r} \\
\Big\downarrow & \nearrow{\scriptstyle \delta_r} & \\
\frac{(R_{i'}^{\circ}\hat{\otimes}_{W(\kappa)}(\mathbb{A}_{\inf}(R,R^+)/\ker(\theta)^r))[\frac{1}{p}]}{\ker(1\otimes\theta)^r}[Y_{n-r+1}, \ldots, Y_n] & &
\end{array}$$

where $\alpha_r(\tilde{X}_l) = Z_l^{1/m} \otimes 1 - 1 \otimes [(Z_l^{1/m})^\flat]$, $\delta_r(Y_l) = \frac{[Z_l^\flat]}{Z_l}$ for all $l > n-r$ and ϵ_r is the natural morphism. Note that δ_r is a surjection. Also the formula $\gamma_r(1 - X_l) = \frac{[Z_l^\flat]}{Z_l}$ tells us that $\frac{[Z_l^\flat]}{Z_l}$ is in the image of γ_r. Therefore to show γ_r is surjective, it suffices to show that α_r is surjective. This just follows from the argument in [Sch16] and is written down below for the sake of completeness of our argument.

First the map

$$(⚂) \qquad \frac{R_{i'}^{\circ}[X_1,\ldots,X_{n-r},\tilde{X}_{n-r+1},\ldots,\tilde{X}_n]}{(X_1,\ldots,X_{n-r},\tilde{X}_{n-r+1},\ldots,\tilde{X}_n)^r} \longrightarrow (R_{i'}^{\circ}\hat{\otimes}_{W(\kappa)}R_{i'}^{\circ})/(\ker\theta_{i'})^r$$

is injective, with cokernel killed by a power of p, where $\theta_{i'}\colon R_{i'}^{\circ}\hat{\otimes}_{W(\kappa)}R_{i'}^{\circ} \to R_{i'}^{\circ}$ is the multiplication map. Here we used the fact that $\mathrm{Sp}(R_{i'}) \to \mathbb{T}^{n-r,r}[\sqrt[m]{Z_l}]$ is étale.

Recall that we have constructed, in step 1, a map $R_{i'}^{\circ} \to \mathbb{B}_r'$ taking values in some open and bounded subring. Composing with the projection onto $\mathbb{B}_{\mathrm{dR}}^+/\ker(\theta)^r$, we see that there is a map $R_{i'} \to \mathbb{B}^+_{\mathrm{dR}}/\ker(\theta)^r$ compatible with θ taking values in some open and bounded subring $\mathbb{B}_{r,0} \subset \mathbb{B}^+_{\mathrm{dR}}/\ker(\theta)^r$ (notice the typo in [Sch16] here). Now we apply $\hat{\otimes}_{R^\circ_{i'}}\mathbb{B}_{r,0}$ to the map (⚁). We get

$$\mathbb{B}_{r,0}[X_1,\ldots,X_{n-r},\tilde{X}_{n-r+1},\ldots,\tilde{X}_n]/(X_1,\ldots,X_{n-r},\tilde{X}_{n-r+1},\ldots,\tilde{X}_n)^r \to (R^\circ_{i'}\hat{\otimes}_{W(\kappa)}\mathbb{B}_{r,0})/((\ker\theta_{i'})^r\hat{\otimes}_{R^\circ_{i'}}\mathbb{B}_{r,0})$$

is an isomorphism up to a bounded power of p. Finally we invert p and use

$$((\ker\theta_{i'})^r\hat{\otimes}_{R^\circ_{i'}}\mathbb{B}_{r,0}) \subset (\ker\theta)^r$$

to conclude that α_r is a surjection. □

Corollary 6.10 (logarithmic Poincaré Lemma)**.** *Let X be a smooth rigid space of dimension n over $\mathrm{Sp}(k)$ with SSNC divisor D. The following sequence of sheaves on X_{prolog} is exact.*

$$0 \to \mathbb{B}^+_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}^+_{\log\mathrm{dR}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}^+_{\log\mathrm{dR}}\otimes_{\mathcal{O}_X}\Omega^1_X(\log D) \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}^+_{\log\mathrm{dR}}\otimes_{\mathcal{O}_X}\Omega^n_X(\log D) \to 0.$$

Moreover, the derivation ∇ satisfies Griffiths transversality with respect to the filtration on $\mathcal{O}\mathbb{B}^+_{\log\mathrm{dR}}$, and with respect to the grading giving $\Omega^i_X(\log D)$ degree i, the sequence is strict exact.

Proof. This follows from Proposition 6.8 and the equation

$$d(X_l) = d(1 - \frac{[Z_l^\flat]}{Z_l}) = \frac{[Z_l^\flat]}{Z_l^2}\,dZ_l = \frac{[Z_l^\flat]}{Z_l}\frac{dZ_l}{Z_l} = (1 - X_l)\cdot d\log(Z_l).$$

□

Remark 6.11. From the above Corollary, especially the strict exactness, we get the following exact sequence

$$0 \to \mathbb{B}_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}_{\log\mathrm{dR}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\log\mathrm{dR}}\otimes_{\mathcal{O}_X}\Omega^1_X(\log D) \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\log\mathrm{dR}}\otimes_{\mathcal{O}_X}\Omega^n_X(\log D) \to 0$$

which share the same properties as the sequence above.

In particular, we get the following short exact sequence, which is due to Faltings in the case of algebraic varieties, see [Fal88, Theorem 4.3].

Corollary 6.12 (Faltings's extension)**.** *Let X be a smooth rigid space over $\mathrm{Sp}(k)$ with SSNC divisor D. Then we have a short exact sequence of sheaves over X_{prolog},*

$$0 \to \hat{\mathcal{O}}_{X_{\log}}(1) \to \mathrm{gr}^1\mathcal{O}\mathbb{B}^+_{\log\mathrm{dR}} \to \hat{\mathcal{O}}_{X_{\log}} \otimes_{\mathcal{O}_{X_{\log}}} \Omega^1_X(\log D) \to 0$$

Corollary 6.13. *Let $X \to \mathbb{T}^{n-r,r}, \tilde{X}, K$ and X_i be as above. For any $i \in \mathbb{Z}$, we have an isomorphism of sheaves over $X_{\mathrm{prolog}}/(X_K, \tilde{X}_K)$,*

$$\mathrm{gr}^i\mathcal{O}\mathbb{B}_{\log\mathrm{dR}} \cong \xi^i\hat{\mathcal{O}}_{X_{\log}}[X_1/\xi,\ldots,X_n/\xi].$$

In particular,

$$\mathrm{gr}^\bullet\mathcal{O}\mathbb{B}_{\log\mathrm{dR}} \cong \hat{\mathcal{O}}_{X_{\log}}[\xi^{\pm 1}, X_1,\ldots,X_n],$$

where ξ and X_i have degree 1.

The following is analogous to [Sch13a, Proposition 6.16].

Proposition 6.14. *Let $X = \mathrm{Spa}(R, R^\circ)$ be an affinoid adic space of finite type over $\mathrm{Spa}(k, k^\circ)$ with an étale map $X \to \mathbb{T}^{n-r,r}$ that factors as a composite of rational embeddings and finite étale maps.*

(1) *Assume that K contains all roots of unity. Then*

$$H^q(X_{K,\mathrm{prolog}}, \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = 0$$

unless $q = 0$, in which case it is $R\hat{\otimes}_k K$.

(2) *We have*

$$H^q(X_{\mathrm{prolog}}, \mathrm{gr}^i\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = 0$$

unless $i = 0$ and $q = 0, 1$. We also have $H^0(X_{\mathrm{prolog}}, \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = R$ and $H^1(X_{\mathrm{prolog}}, \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = R\log\chi$. Here $\chi\colon \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}_p^\times$ is the cyclotomic character and

$$\log\chi \in \mathrm{Hom}_{cont}(\mathrm{Gal}(\bar{k}/k), \mathbb{Q}_p) = H^1_{cont}(\mathrm{Gal}(\bar{k}/k), \mathbb{Q}_p)$$

is its logarithm.

Proof. (1) As before, denote $X_K \times_{\mathbb{T}_K^{n-r,r}} \tilde{\mathbb{T}}_K^{n-r,r} =: \tilde{X}_K$ where $\widehat{\tilde{X}_K} = \mathrm{Spa}(\hat{R}, \hat{R}^\circ)$. We see that $\tilde{X}_K \to X_K$ is a $\mathbb{Z}_p^{n-r} \times \hat{\mathbb{Z}}^r$-cover and all multiple-fold fibre products of $\tilde{X}_K$ over X_K are affinoid perfectoid. By Corollary 6.13 and Remark 6.3 we see that all higher cohomology groups of the sheaves considered vanish and

$$H^q(X_{K,\mathrm{prolog}}, \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = H^q_{cont}(\mathbb{Z}_p^{n-r} \times \hat{\mathbb{Z}}^r, \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}(\tilde{X}_K)).$$

Note that we may write

$$\mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}(\tilde{X}_K) = \hat{R}[V_1, \ldots, V_n],$$

where $V_i = t^{-1}\log(\frac{[T_i^\flat]}{T_i})$ and $t = \log([\epsilon])$. Let γ_i be the i-th basis vector of $\mathbb{Z}_p^{n-r} \times \hat{\mathbb{Z}}^r$, then we have (c.f. [Sch13a, Lemma 6.17])

$$\gamma_i(V_j) = V_j + \delta_{ij}.$$

Next we claim the inclusion

$$(R\hat{\otimes}_k K)[V_1, \ldots, V_n] \subset \hat{R}[V_1, \ldots, V_n]$$

induces an isomorphism on the continuous group cohomologies. This can be seen via checking the graded pieces given by the degree of polynomials. On the gradeds the group action on V_i's is trivial, therefore it suffices to check that $R\hat{\otimes}_k K \subset \hat{R}$ induces an isomorphism on continuous group cohomologies. This just follows from Lemma 4.5(2) and Lemma 5.5, c.f. [Sch13a, Lemma 6.18].

Lastly we need to compute

$$H^q_{cont}(\mathbb{Z}_p^{n-r} \times \hat{\mathbb{Z}}^r, (R\hat{\otimes}_k K)[V_1, \ldots, V_n]).$$

But since all the factors $\hat{\mathbb{Z}}_{(p)} \coloneqq \prod_{l \neq p} \mathbb{Z}_l$ acts trivially on $(R\hat{\otimes}_k K)[V_1, \ldots, V_n]$ which has p-adic topology, we see that the continuous group cohomology is the same as

$$H^q_{cont}(\mathbb{Z}_p^n, (R\hat{\otimes}_k K)[V_1, \ldots, V_n]).$$

Now the last paragraph of the proof of [Sch13a, Proposition 6.16(i)] shows that these cohomology groups are 0 whenever $q > 0$ and is equal to $R\hat{\otimes}_k K$ when $q = 0$.

(2) Let k' be the completion of $\cup_{p\nmid n} k(\mu_n)$ and take K as the completion of $k'(\mu_{p^\infty})$. Also let us denote $G = \mathrm{Gal}(k(\mu_\infty)/k) = H \times \Gamma$ where $H = \mathrm{Gal}((\cup_{p\nmid n} k(\mu_n))/k)$ and $\Gamma = \mathrm{Gal}(k(\mu_{p^\infty})/k)$. By the same argument as in the proof of [Sch13a, Proposition 6.16(ii)], we see that

$$H^q(X_{\mathrm{prolog}}, \mathrm{gr}^i\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = H^q_{cont}(G, R\hat{\otimes}_k K(i))$$

and

$$H^q_{cont}(\Gamma, R\hat{\otimes}_k K(i)) = R_{k'} \otimes_{\mathbb{Q}_p} H^q_{cont}(\Gamma, \mathbb{Q}_p(i))$$

and the latter is well-known, see [Tat67]. Moreover we know that the action of H on $\log\chi$ is trivial and

$$H^q_{cont}(H, R_{k'}) = 0$$

unless $q = 0$ in which case it is R. Indeed, since H is a profinite group, we know that $H^q_{cont}(H, R_{k'}) = (H^q_{cont}(H, R^\circ\hat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{k'}))[1/p]$. Now it suffices to show $H^q_{cont}(H, R^\circ\hat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{k'}) = 0$ for all $q > 0$, and $H^0_{cont}(H, R^\circ\hat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{k'}) = R^\circ$. We claim that $H^q_{cont}(H, (R^\circ\hat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{k'})/\varpi^m) = 0$ for all $m > 0$, unless $q = 0$ in which case it is given by R°/ϖ^m. To prove this claim we simply notice that by induction on m and the fact that $R^\circ\hat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{k'}$ is ϖ-torsion free, it suffices to prove it when $m = 1$ which follows from Hilbert 90. The above claim yields that $H^q_{cont}(H, R^\circ\hat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{k'}) = R^q\varprojlim_m R^\circ/\varpi^m$, which easily implies what we want.

Put all these together along with Hochschild–Serre spectral sequence yields the results we want. □

Corollary 6.15. *Let X be a smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$ with an SSNC divisor D. Let i, j be two integers and let m be a positive integer, then we have*

(1) $R^q\nu_*(\mathrm{Fil}^i\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}/\mathrm{Fil}^{i+m}\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = 0$ *unless* $q = 0, 1$ *and* $0 \in [i, i+m)$, *in which case* $R^0\nu_*$ *is given by* $\mathcal{O}_{X_{\log}}$ *and* $R^1\nu_*$ *is given by* $\mathcal{O}_{X_{\log}} \cdot \log\chi$.

(2) $R^q\nu_*\mathrm{Fil}^i\mathcal{O}\mathbb{B}_{\log\mathrm{dR}} = 0$ *unless* $q = 0, 1$ *and* $i \leq 0$ *in which case* $R^0\nu_*$ *is given by* $\mathcal{O}_{X_{\log}}$ *and* $R^1\nu_*$ *is given by* $\mathcal{O}_{X_{\log}}\log\chi$. *The above computation also holds for* $i = -\infty$ *where* $\mathrm{Fil}^{-\infty}\mathcal{O}\mathbb{B}_{\log\mathrm{dR}} = \mathcal{O}\mathbb{B}_{\log\mathrm{dR}}$.

(3) $R^i\nu_*\hat{\mathcal{O}}_{X_{\log}}(j) = 0$ *unless*
   - $i = j$ *in which case it is given by* $\Omega^j_{X_{\log}}(\log D)$ *or;*
   - $i = j+1$ *in which case it is given by* $\Omega^j_{X_{\log}}(\log D) \cdot \log\chi$.[5]

   Moreover the isomorphism $R^1\nu_*\hat{\mathcal{O}}_{X_{\log}}(1) \cong \Omega^1_{X_{\log}}(\log D)$ *is given by the Faltings's extension (c.f. Corollary 6.12).*

Proof. (1) trivially follows from Proposition 6.14(2).

(2) follows from (1) by commuting limit and colimit with cohomology.

(3) follows from applying $R\nu_*$ to j-th graded piece of Remark 6.11 which reads

$$0 \to \hat{\mathcal{O}}_{X_{\log}}(j) \to \mathrm{gr}^j\mathcal{O}\mathbb{B}_{\log\mathrm{dR}} \to \mathrm{gr}^{j-1}\mathcal{O}\mathbb{B}_{\log\mathrm{dR}} \otimes_{\mathcal{O}_{X_{\log}}} \Omega^1_{X_{\log}}(\log D) \to \cdots.$$

The last statement can be seen via the natural morphism from the sequence in Corollary 6.12 to the above sequence where $j = 1$. □

Remark 6.16. Let $X = \mathrm{Sp}(R) \xrightarrow{f} \mathbb{T}^{n-r,r}$ be as in Proposition 6.14. Denote $f^*(T_l)$ by f_l where $l > n-r$. Then by the same argument, one can show that

$$H^q((X, X[\sqrt[m]{f_l}]_{\bar{k}}), \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\log\mathrm{dR}}) = 0$$

unless $q = 0$, in which case it is $R[\sqrt[m]{f_l}]\hat{\otimes}_k\hat{\bar{k}}$.

[5] Note the typo in [Sch13a, Remark 6.20].

Remark 6.17. Let X be a smooth adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$ where C is an algebraically closed non-archimedean extension of $\mathbb{Q}_p$. Similar as in [Sch13b, Proposition 3.23 and Lemma 3.24], one can show that there is a commutative diagram

$$\begin{array}{ccc} \bigwedge^k (R^1\nu_*\hat{\mathcal{O}}_{X_{\log}}(1)) & \longrightarrow & R^k\nu_*\hat{\mathcal{O}}_{X_{\log}}(k) \\ \downarrow \cong & & \downarrow \cong \\ \bigwedge^k (\Omega^1_{X_{\log}}(\log D)) & \longrightarrow & \Omega^k_{X_{\log}}(\log D) \end{array}$$

where the vertical maps are obtained in the same fashion as above.

7. Comparisons

7.1. Vector bundles on $X_{\log}$.

Definition 7.1. A *vector bundle* $\mathcal{F}$ on $X_{\log}$ is a sheaf of $\mathcal{O}_{X_{\log}}$-modules such that there exists a finite affinoid covering $(V_i, N_i) \to (X, X)$ and finite projective $\Gamma(N_i, \mathcal{O}_{N_i})$-modules M_i with isomorphism

$$\mathcal{F}|_{(V_i,N_i)} \cong M_i \otimes_{\Gamma(N_i,\mathcal{O}_{N_i})} \mathcal{O}_{X_{\log}}.$$

Here $(M_i \otimes_{\Gamma(N_i,\mathcal{O}_{N_i})} \mathcal{O}_{X_{\log}})(W, M) := M_i \otimes_{\Gamma(N_i,\mathcal{O}_{N_i})} \Gamma(M, \mathcal{O}_M)$ for any object (W, M) over (V_i, N_i), and by affinoid covering we mean a covering with all V_i (hence N_i) being affinoid.

Remark 7.2. Note that since M_i's are assumed to be finite projective, they are direct summand in finite free modules. Therefore $M_i \otimes_{\mathcal{O}(N)} \mathcal{O}_{X_{\log}}$ indeed defines a sheaf on the localized site $X_{\log}/(V, N)$. We say $\mathcal{F}$ is represented by a finite projective module on $(V, N) \in X_{\log}$ if one can find an M and an isomorphism as in the previous definition.

Theorem 7.3 (Theorem A)**.** *Let $\mathcal{F}$ be a vector bundle on $X_{\log}$. Then there exists a positive integer m such that for any affinoid $V \xrightarrow{f} X$ étale over X with $f^{-1}(D_l)$ being defined by f_l (where D_l is the l-th component of D), there exists a finite projective $\mathcal{O}(V[\sqrt[m]{f_l}])$-module M and an isomorphism*

$$\mathcal{F}|_{(V,V[\sqrt[m]{f_l}])} \cong M \otimes_{\mathcal{O}(V[\sqrt[m]{f_l}])} \mathcal{O}_{X_{\log}}.$$

Proof. Let V_i and N_i be as in the definition, by passing to refinement we may assume the preimage of D_l in V_i is defined by a single function $f_{i.l}$. Then by Theorem 2.2 we can find a positive integer m such that $N_i[\sqrt[m]{f_{i,l}}] \to V_i[\sqrt[m]{f_{i,l}}]$ is finite étale. Consider the following diagram:

$$\begin{array}{ccccc} \coprod_i (V_i', N_i') & = & \coprod_i \left((V_i, N_i) \times_{(X,X)} (V, V[\sqrt[m]{f_l}]) \right) & \longrightarrow & (V, V[\sqrt[m]{f_l}]) \\ & & \downarrow & & \downarrow \\ & & \coprod_i (V_i, N_i) & \longrightarrow & (X, X). \end{array}$$

From the diagram and our choice of m, we see that $\coprod_i N_i' \to V[\sqrt[m]{f_l}]$ is an étale covering in the usual sense in rigid geometry and our sheaf $\mathcal{F}$ is represented by finite projective modules M_i on (V_i', N_i'). Therefore étale descent implies what we want. $\square$

Theorem 7.4 (Theorem B)**.** *For any vector bundle $\mathcal{F}$ and any affinoid $(V, N) \in X_{\log}$, assume one of the following conditions holds*

(1) *$\mathcal{F}|_{(V,N)}$ is represented by a finite projective $\mathcal{O}(N)$-module M or;*
(2) *preimage of D_l in V is defined by a single function f_l for all l,*

then we have

$$H^q((V, N), \mathcal{F}) = 0$$

for all $q > 0$.

Proof. We first observe that the statement of this theorem for objects satisfying condition (2) implies the statement for objects satisfying condition (1). Indeed, we can cover V by V_i satisfying (2). Therefore by the statement for objects satisfying condition (2), we see that $(V_i, V_i \times_V N) \to (V, N)$ is an acyclic cover for $\mathcal{F}$. Hence by Čech-to-cohomology spectral sequence we see that $H^q((V, N), \mathcal{F})$ is the same as q-th Čech cohomology for this covering, which is the cohomology of Čech complex associated to the affinoid covering $\{V_i \times_V N\}$ for our finite projective module M. Hence we get $H^q((V, N), \mathcal{F}) = 0$ as N is an affinoid.

From now on we will assume that our (V, N) satisfies condition (2). We will prove the vanishing of cohomology by induction on q (the starting case $q = 1$ follows from the same argument), therefore we will assume for objects satisfying (2) the cohomology of $\mathcal{F}$ vanishes up to degree $q - 1$.

Let $\xi \in H^q((V, N), \mathcal{F})$ be a cohomology class. Then there exists a covering by qcqs objects $(V', N') \to (V, N)$ such that ξ pulls back to zero in $H^q((V', N'), \mathcal{F})$. Then by Theorem 2.2 and Theorem 7.3 we can find an m such that

(1) $\mathcal{F}|_{(V,N[\sqrt[m]{f_l}])}$ is represented by a finite projective $\mathcal{O}(N[\sqrt[m]{f_l}])$-module M;
(2) $N'' =: (N[\sqrt[m]{f_l}] \times_N N')^\nu \to N[\sqrt[m]{f_l}]$ is an étale covering.

Let $k' = k[\zeta_m]$ where ζ_m is a primitive m-th root of unity. Let us consider the following diagram

$$\begin{array}{ccc} (V, N[\sqrt[m]{f_l}]_{k'}) & \xleftarrow{\ \beta\ } & (V', N''_{k'}) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow \\ (V, N) & \longleftarrow & (V', N') \end{array}$$

where subscript $(\cdot)_{k'}$ means the base change of spaces from k to k'. The cohomology class ξ is assumed to be zero on (V', N'), hence it is zero on $(V', N''_{k'})$. Now by Čech-to-cohomology spectral sequence

$$E_2^{a,b} = \check{\mathrm{H}}^a(\beta, \underline{H}^b\mathcal{F}) \Longrightarrow H^{a+b}((V, N[\sqrt[m]{f_l}]_{k'}), \mathcal{F})$$

and induction hypotheses, we have an exact sequence as follows

$$0 \to \check{\mathrm{H}}^q(\beta, \mathcal{F}) \to H^q((V, N[\sqrt[m]{f_l}]_{k'}), \mathcal{F}) \to \mathrm{H}^q((V', N''_{k'}), \mathcal{F}).$$

From this sequence, we see that $\xi_{(V,N[\sqrt[m]{f_l}]_{k'})}$ is represented by a class in Čech cohomology of $\mathcal{F}$ associated to the cover given by β. Moreover, for $q \geq 1$, we have that $\check{\mathrm{H}}^q(\beta, \mathcal{F})$ is zero by (1), (2) and étale descent. It follows that $\xi_{(V,N[\sqrt[m]{f_l}]_{k'})} = 0$.

Therefore, as above, by induction hypothese and Čech-to-cohomology spectral sequence we see that ξ is represented by a class in $\check{H}^q(\alpha, \mathcal{F})$, the Čech cohomology of $\mathcal{F}$ associated to the cover given by α. Now we notice that the j-th fold product of $(V, N[\sqrt[m]{f_l}]_{k'})$ over (V, N) is isomorphic to $(V, N[\sqrt[m]{f_l}]_{k'}) \times G \times \ldots \times G$ with

$(j-1)$-st copies of G's appearing in the product. Here G is the Galois group of the covering $N[\sqrt[m]{f_l}]_{k'}$ over N. The sheaf condition gives us an action of G on M and $\check{H}^q(\alpha, \mathcal{F}) = H^q(G, M)$ which is zero because M is divisible and G is a finite group. This proves that $\xi = 0$. □

Corollary 7.5. *Let $\lambda\colon X_{\log} \to X_{\text{ét}}$ be the morphism of sites sending U to (U, U). Then we have*

$$R\lambda_*\mathcal{O}_{X_{\log}} \cong \mathcal{O}_{X_{\text{ét}}}$$

Therefore for any vector bundle $\mathcal{F}$ on $X_{\text{ét}}$, we have

$$\mathcal{F} \cong R\lambda_*\lambda^*\mathcal{F}.$$

In particular, we see that $\lambda^*(\cdot)$ gives a fully faithful embedding from the category of vector bundles on $X_{\text{ét}}$ to that on $X_{\log}$.

Proof. The first assertion follows from Theorem A and B above. The second assertion follows from adjunction formula. □

Theorem 7.6. *For any vector bundle $\mathcal{F}$ on $X_{\log}$, the cohomology groups*

$$H^q((X, X), \mathcal{F})$$

are finite dimensional k vector spaces for all q.

Proof. By Lemma 5.4 we may find two affinoid coverings $\{V_i\}$ and $\{V_i'\}$ of X, such that

(1) $V_i' \Subset_X V_i$ for all i;
(2) V_i (hence V_i') satisfies condition (2) in Theorem B, i.e., $D_l \cap V_i$ is given by vanishing of $f_{i,l}$.

Now by Theorem A, there is an m such that $\mathcal{F}|_{(V_i, V_i[\sqrt[m]{f_{i,l}}])}$ is represented by a finite projective module M_i. By the same reasoning $\mathcal{F}|_{(V_i', V_i'[\sqrt[m]{f_{i,l}}])}$ is represented by $M_i|_{V_i'[\sqrt[m]{f_{i,l}}]}$. By Theorem B, we see that the covering $\coprod_i (V_i, V_i[\sqrt[m]{f_{i,l}}]) \to (X, X)$ (resp. $\coprod_i (V_i', V_i'[\sqrt[m]{f_{i,l}}]) \to (X, X)$) is acyclic for $\mathcal{F}$. Therefore we see that

$$\begin{aligned} H^q((X, X), \mathcal{F}) &= \check{H}^q(\coprod_i (V_i, V_i[\sqrt[m]{f_{i,l}}]) \to (X, X), \mathcal{F}) \\ &= \check{H}^q(\coprod_i (V_i', V_i'[\sqrt[m]{f_{i,l}}]) \to (X, X), \mathcal{F}). \end{aligned}$$

On the other hand, by our choice of V_i and V_i', we have that $V_i'[\sqrt[m]{f_{i,l}}]$ is strictly contained in $V_i[\sqrt[m]{f_{i,l}}]$. Therefore the map from the Čech complex of $(V_i, V_i[\sqrt[m]{f_{i,l}}])$ to that of $(V_i', V_i'[\sqrt[m]{f_{i,l}}])$ is strictly continuous and an isomorphism on cohomology groups. Hence we see that these cohomology groups are finite dimensional k vector spaces. See also the proof of Kiehl's proper mapping theorem in [Bos14, 6.4]. □

The above theorem implies the following base change lemma, which will be used later.

Lemma 7.7. *Let X be a smooth adic space over* $\mathrm{Spa}(k, \mathcal{O}_k)$ *with an SSNC divisor D. Let $\mathcal{A}$ be a vector bundle on $X_{\log}$. Then for all $i, j \in \mathbb{Z}$, we have an isomorphism*

$$H^j((X, X), \mathcal{A}) \otimes_k \mathrm{gr}^i B_{\mathrm{dR}} \cong H^j((X, X_{\bar{k}}), \mathcal{A} \otimes_{\mathcal{O}_{X_{\log}}} \mathrm{gr}^i \mathcal{O}\mathbb{B}_{\mathrm{logdR}})$$

where the latter group is computed on X_{prolog}.

Proof. By twisting, it suffices to prove the case where $i = 0$. The statement reads

$$H^j((X,X),\mathcal{A})\otimes_k \hat{\bar{k}} \cong H^j((X,X_{\bar{k}}),\mathcal{A}\otimes_{\mathcal{O}_{X_{\log}}} \mathrm{gr}^0\mathcal{O}\mathbb{B}_{\mathrm{logdR}}).$$

To this end, let $\coprod(V_i, V_i[\sqrt[m]{f_{i,l}}])$ be an acyclic covering of $\mathcal{A}$ as in the proof of Theorem 7.6. Denote the Čech complex associated to $\mathcal{A}$ and this covering by $\mathcal{C}^\bullet$. By Remark 6.16, we know that RHS is cohomology groups of $\mathcal{C}^\bullet\hat{\otimes}_k\hat{\bar{k}}$. Therefore we reduce to the statement

$$H^j(\mathcal{C}^\bullet)\hat{\otimes}_k\hat{\bar{k}} \cong H^j(\mathcal{C}^\bullet\hat{\otimes}_k\hat{\bar{k}}).$$

This follows from the fact that $\mathcal{C}^\bullet$ has finite dimensional (as k vector spaces) cohomology groups. □

Remark 7.8. (1) There are interesting vector bundles on $X_{\log}$ not coming from $X_{\text{ét}}$. Assume $D \subset X$ is a smooth divisor, the "square root" of the ideal sheaf of D, given by $\sqrt{I_D}(V,N) = \{a \in \Gamma(N,\mathcal{O}_N) | a^2 \in g^*I_{(D)}\}$ (for $g\colon N \to V$), is such an example.

(2) One can develop a more general theory of "coherent" sheaf and prove similar theorems as above for these sheaves. We will not work it out in this note however, since it is irrelevant to the theme of this note.

7.2. **Proof of the Comparison.** In this subsection, let k be an discretely valued complete non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field κ. Let X be a smooth adic space over $\mathrm{Spa}(k,\mathcal{O}_k)$ with an SSNC divisor D. Denote $X \setminus D$ by U. Denote an algebraic closure of k by $\bar{k}$ and its completion by $\hat{\bar{k}}$. Let $A_{\inf}$, $B_{\inf}$, etc. be the period rings as defined by Fontaine.

Theorem 7.9. *There is a canonical isomorphism*

$$H^m\big((X,X_{\bar{k}}),\mathbb{B}^+_{\mathrm{dR}}\big)\otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}} \cong H^m\big(X,\Omega^\bullet_{X_{\log}}(\log D)\big)\otimes_k B_{\mathrm{dR}}$$

compatible with filtrations and $\mathrm{Gal}(\bar{k}/k)$*-actions.*

Moreover, we have a $\mathrm{Gal}(\bar{k}/k)$*-equivariant isomorphism*

$$H^m\big((X,X_{\bar{k}}),\hat{\mathcal{O}}_{X_{\log}}\big) \cong \bigoplus_{a+b=m} H^a(X,\Omega^b_X(\log D))\otimes_k \hat{\bar{k}}(-b).$$

Remark 7.10. By Corollary 7.5, we have canonical isomorphisms:

$$H^m\big(X,\Omega^\bullet_{X_{\log}}(\log D)\big) \cong H^m\big((X,X),\Omega^\bullet_{X_{\log}}(\log D)\big)$$

and

$$H^a(X,\Omega^b_X(\log D)) \cong H^a((X,X),\Omega^b_X(\log D)),$$

where the left hand side denotes the cohomology computed on the rigid space X and the right hand side denotes the cohomology computed on the Faltings site $X_{\log}$.

Proof. In the filtered derived category we have

$$R\Gamma\big((X,X_{\bar{k}}),\mathbb{B}^+_{\mathrm{dR}}\big)\otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}} = R\Gamma\big((X,X_{\bar{k}}),\mathbb{B}_{\mathrm{dR}}\big) = R\Gamma\big((X,X_{\bar{k}}),\mathcal{O}\mathbb{B}_{\mathrm{logdR}}\otimes_{\mathcal{O}_{X_{\log}}}\Omega^\bullet_{X_{\log}}(\log D)\big)$$

where the second equality follows from Poincaré Lemma (c.f. Remark 6.11). We claim that the natural map of filtered complexes

$$\Omega^\bullet_{X_{\log}}(\log D) \to \mathcal{O}\mathbb{B}_{\mathrm{logdR}}\otimes_{\mathcal{O}_{X_{\log}}}\Omega^\bullet_{X_{\log}}(\log D)$$

induces a quasi-isomorphism

$$R\Gamma\big((X,X),\Omega^\bullet_{X_{\log}}(\log D)\big)\otimes_k \mathrm{B}_{\mathrm{dR}} \to R\Gamma\big((X,X_{\bar{k}}),\mathcal{O}\mathbb{B}_{\mathrm{logdR}}\otimes_{\mathcal{O}_{X_{\log}}}\Omega^\bullet_{X_{\log}}(\log D)\big).$$

It suffices to check the claim above on graded pieces. Further filtering by using naive filtration of $\Omega^\bullet_{X_{\log}}(\log D)$, one is reduced to show that for any vector bundle $\mathcal{A}$ on $X_{\log}$ and $i \in \mathbb{Z}$, the map

$$R\Gamma\big((X,X),\mathcal{A}\big) \otimes_k \mathrm{gr}^i \mathrm{B}_{\mathrm{dR}} \to R\Gamma\big((X,X_{\bar{k}}),\mathcal{A} \otimes_{\mathcal{O}_{X_{\log}}} \mathrm{gr}^i \mathcal{O}\mathbb{B}_{\mathrm{logdR}}\big)$$

is a quasi-isomorphism. This follows from Lemma 7.7.

Therefore we have constructed a quasi-isomorphism

$$R\Gamma\big((X,X),\Omega^\bullet_{X_{\log}}(\log D)\big) \otimes_k \mathrm{B}_{\mathrm{dR}} \to R\Gamma\big((X,X_{\bar{k}}),\mathbb{B}^+_{\mathrm{dR}}\big) \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$$

in filtered derived category. Now we get comparison results, by taking cohomology of both sides (resp. of the 0-th graded piece of both sides). □

Let us make a remark about the notion of local systems on sites $X_{\log}$ and X_{prolog}.

Remark 7.11. Note that for any $\mathbb{Z}/p^n$-local system $\mathbb{L}_n$ on U, $(u_{X,*}\mathbb{L}_n)(V,N) = \mathbb{L}_n(N^\circ)$ for any $(V,N) \in X_{\log}$ (see Theorem 2.8). By Lemma 3.9(1), for any $(V, N = \varprojlim N_i) \in X_{\mathrm{prolog}}$ and any $i \geq 0$ we have

$$H^i((V,N),\nu^*(u_{X,*}\mathbb{L}_n)) = \varinjlim_i H^i((N_i^\circ),\mathbb{L}_n).$$

If no confusion shall arise, we will still denote $u_{X,*}\mathbb{L}_n$ (resp. $\nu^*(u_{X,*}\mathbb{L}_n)$) by $\mathbb{L}_n$.

Recall the notion of lisse $\mathbb{Z}_p$-sheaf as in [Sch13a, Definition 8.1]. Analogously, we make the following definition.

Definition 7.12. Let $\hat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^n$ as sheaves on X_{prolog}. Then a *lisse $\hat{\mathbb{Z}}_p$-sheaf* on X_{prolog} is a sheaf $\mathbb{L}$ of $\hat{\mathbb{Z}}_p$-modules on X_{prolog}, such that locally $\mathbb{L}$ is isomorphic to $\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M$, where M is a finitely generated $\mathbb{Z}_p$-module.

In concrete terms, $\mathbb{L}$ is a lisse $\hat{\mathbb{Z}}_p$-sheaf just means that there is a covering $\coprod_j (V,N)_j \to (X,X)$ in X_{prolog} such that for each j there is a finitely generated $\mathbb{Z}_p$-module M_j and a (non-canonical)-isomorphism

$$\mathbb{L}|_{(V,N)_j} \simeq (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M_j)|_{(V,N)_j} := \varprojlim_m (\nu^*(u_{X,*}M_j/p^m))|_{(V,N)_j}.$$

Note that if X is connected, then all the M_j's are automatically isomorphic to each other as finitely generated $\mathbb{Z}_p$-modules.

Proposition 7.13. *Let $\mathbb{L}_\bullet$ be a lisse $\mathbb{Z}_p$-sheaf on $U_{\text{ét}}$. Then $\mathbb{L} = \varprojlim \nu^*(u_{X,*}\mathbb{L}_m)$ is a lisse sheaf of $\hat{\mathbb{Z}}_p$-modules on X_{prolog}. This functor gives an equivalence of categories. Moreover, $R^j \varprojlim \nu^*(u_{X,*}\mathbb{L}_m) = 0$ for $j > 0$.*

Proof. Without loss of generality, let us assume that X is connected. First notice that there exists a system of finite étale covers $\{U_m\}$ to U and a compatible system of isomorphisms $\mathbb{L}_m|_{U_m} \xrightarrow{\simeq} (M/p^m)|_{U_m}$ where M is a finitely generated $\mathbb{Z}_p$-module. By [Han17, Theorem 1.6], each U_m extends to an $N_m \to X$. Let $N = \varprojlim N_m$, then $(X,N) \to (X,X)$ is a covering in X_{prolog}. We see that, with $\mathbb{L}$ as defined in this proposition, we have an isomorphism $\mathbb{L}|_{(X,N)} \simeq (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M)|_{(X,N)}$. Hence $\mathbb{L}$ as defined in this proposition is a lisse sheaf of $\hat{\mathbb{Z}}_p$-modules on X_{prolog}. Conversely, let $\mathbb{L}$ be a lisse sheaf of $\hat{\mathbb{Z}}_p$-modules. Let $(V,N)_j$ and $M_j = M$ be as in the discussion before this proposition. Then we see that $\mathbb{L}|_{N_j^\circ} \simeq (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M_j)|_{N_j^\circ}$ gives rise to a lisse sheaf of $\hat{\mathbb{Z}}_p$-modules on $U_{\text{proét}}$, here each $N_j = \varprojlim N_{j,l}$ is a pro-object in $V_{\text{profét}}$ and

$N_j^\circ := \varprojlim N_{j,l}^\circ$ naturally is an object in $U_{\text{proét}}$. Therefore by [Sch13a, Proposition 8.2], we get back a lisse $\mathbb{Z}_p$-sheaf on $U_{\text{ét}}$. One verifies that this construction is an inverse to the functor described in this proposition, therefore the two categories are equivalent under $\mathbb{L}_\bullet \mapsto \mathbb{L} = \varprojlim \nu^*(u_{X,*}\mathbb{L}_n)$.

To check that $R^j \varprojlim \nu^*(u_{X,*}\mathbb{L}_n) = 0$, we verify the conditions in [Sch13a, Lemma 3.18] for $\mathcal{F}_m = \mathbb{L}_m$. The condition (i) of [Sch13a, Lemma 3.18] trivially follows from the fact that $\mathbb{L}_m$ takes value in finite abelian groups. The condition (ii) of [Sch13a, Lemma 3.18] follows from Proposition 2.12, Theorem 2.8, Lemma 3.9(1) and [Kie67, Theorem 1.18]. Indeed, [Kie67, Theorem 1.18] tells us that there is an open cover $\{V_i\}$ of X with each V_i of the form $S \times \mathbb{D}^r$ (and $D \cap V_i = S \times \Delta$) as in Proposition 2.12. Now we take N_i to be the pro-system of all $N_{i,l}$ over V_i. By Theorem 2.8 and Lemma 3.9(1), we have $H^j((V_i, N_i), \mathbb{L}_m) = \varinjlim_l H^j(N_{i,l}^\circ, \mathbb{L}_m)$ which is zero by Proposition 2.12. □

In this note, we will only consider the case where $\mathbb{L}_m = \mathbb{Z}/p^m$.

Theorem 7.14. *We have a natural* $\mathrm{Gal}(\bar{k}/k)$*-equivariant isomorphism*

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B^+_{\text{dR}} \cong H^i((X, X_{\bar{k}}), \mathbb{B}^+_{\text{dR}}).$$

Remark 7.15. Here by $H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p)$ we mean $\varprojlim_m H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}/p^m)$. Note that U is the complement of an SSNC divisor in a proper smooth adic spaces. It is easy to check that $H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}/p^m) = \varinjlim_{l/k} H^i_{\text{ét}}(U_l, \mathbb{Z}/p^m)$ where the left hand side is understood as the étale cohomology of $\mathbb{Z}/p^m$ on the adic space $U_{\bar{k}}$ and the colimit on the right hand side is taking over finite field extensions l of k.

It follows from Theorem 2.8 that

$$H^i(U_l, \mathbb{Z}/p^m) = H^i((X, X_l), \mathbb{Z}/p^m).$$

By Remark 7.11, we can take the colimit over finite field extensions l of k and get

$$H^i(U_{\overline{k}}, \mathbb{Z}/p^m) = H^i((X, X_{\overline{k}}), \mathbb{Z}/p^m).$$

Taking inverse limit over m, we have

$$\varprojlim_m H^i(U_{\overline{k}}, \mathbb{Z}/p^m) = R\varprojlim_m H^i((X, X_{\overline{k}}), \mathbb{Z}/p^m) = H^i((X, X_{\overline{k}}), \varprojlim_m \mathbb{Z}/p^m)$$

where the first identity is due to the finiteness of $H^i((X, X_{\overline{k}}), \mathbb{Z}/p^m)$ (Remark 6.3) and the second identity is due to Proposition 7.13 and the fact that $R\varprojlim$ and $R\Gamma((X, X_{\overline{k}}), -)$ commutes. Therefore, we have

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \cong H^i((X, X_{\bar{k}}), \hat{\mathbb{Z}}_p).$$

Before we start the proof of Theorem 7.14, we need a preliminary discussion on A-R p-adic projective systems, c.f. [Fu11, 10.1].

Lemma 7.16. *Let* $\mathbb{L} = \varprojlim \nu^*(u_{X,*}\mathbb{L}_m)$ *be a lisse sheaf of* $\hat{\mathbb{Z}}_p$*-modules on* X_{prolog}*. Let* H_m *be the cohomology group* $H^i(U_{\overline{k}}, \mathbb{L}_m) = H^i((X, X_{\overline{k}}), \mathbb{L}_m)$*. Then the system* $(H_m)_{m \in \mathbb{N}}$ *is A-R p-adic.*

Proof. The proof is similar to the case of schemes. We may assume that the inverse system $\mathbb{L}^\bullet$ satisfies $\mathbb{L}_{m+1}/p^m \cong \mathbb{L}_m$. We apply results in the theory of l-adic systems to prove this lemma. In fact, we denote $R\Gamma(U_{\overline{k}}, \mathbb{L}_m)$ by $K_m^\bullet$. We claim that the natural maps

$$u_n : K_{n+1}^\bullet \otimes^L_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \xrightarrow{\cong} K_n^\bullet \tag{⚃}$$

are isomorphisms in the derived category. Note that $H^j(K_n^\bullet) = H^j(U_{\overline{k}}, \mathbb{L}_n)$ is zero if $j \notin [0, 2\dim(X)]$. Represent each $K_n^\bullet$ by a bounded above complex of flat $\mathbb{Z}/p^n$-modules with $K_n^j = 0$ for $j > 2\dim(X)$. Moreover, the complex $\ldots \to K_n^{-1} \to K_n^0 \to 0$ is a resolution of $\mathrm{coker}(K_n^{-1} \to K_n^0)$ by flat $\mathbb{Z}/p^n$-modules. It follows that

$$\mathrm{Tor}^i_{\mathbb{Z}/p^n}\left((\mathrm{coker}(K_n^{-1} \to K_n^0), \mathbb{Z}/p\right) = H^{-i}(K_n^\bullet \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p) = H^{-i}(K_1^\bullet) = 0$$

for $i > 0$ where we use the fact that $K_n^\bullet \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p \cong K_1^\bullet$. Therefore, by the local flatness criterion [Mat86, Theorem 22.3], we conclude that $\mathrm{coker}(K_n^{-1} \to K_n^0)$ is a flat $\mathbb{Z}/p^n$-modules. It follows that the complex $K_n^\bullet$ is quasi-isomorphic to the bounded complex of flat $\mathbb{Z}/p^n$-modules

$$0 \to \mathrm{coker}(K_n^{-1} \to K_n^0) \to K_n^1 \to \ldots \to K_n^{2\dim(X)} \to 0.$$

By [Fu11, Lemma 10.1.14], each complex $K_n^\bullet$ is isomorphic in the derived category to a complex $L_n^\bullet$ of free $\mathbb{Z}/p^n$-modules of finite ranks with $L_n^j = 0$ for $j \notin [0, 2\dim(X)]$. The natural isomorphism u_n gives an isomorphism

$$v_n : L_{n+1}^\bullet \otimes^L_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \xrightarrow{\cong} L_n^\bullet$$

in the derived category. By [Fu11, Lemma 10.1.13], this isomorphism v_n is induced by a quasi-isomorphism $L_{n+1}^\bullet \otimes^L_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \xrightarrow{\cong} L_n^\bullet$ of complexes. We apply [Fu11, Proposition 10.1.15] to the system $(L_m^\bullet)_{m\in\mathbb{Z}}$ and show that $H^i(L_m^\bullet) = H_m$ is A-R p-adic.

We give a proof of our claim as follows.

Lemma 7.17. *The natural morphism u_n (see ⚃) is an isomorphism in the derived category.*

Proof. Take an injective resolution of the $\mathbb{Z}/p^m$-modules

$$\mathbb{L}_m \xrightarrow{qis} I^0 \to I^1 \to \ldots.$$

Note that $H^j(K_m^\bullet) = H^j(U_{\overline{k}}, \mathbb{L}_m)$ is zero if $j \notin [0, 2\dim(X)]$. The truncated complex $I'^\bullet$

$$I^0 \to \ldots \to \mathrm{Im}(I^{2\dim(X)-1} \to I^{2\dim(X)}) \to 0$$

is an $R\Gamma(U_{\overline{k}}, -)$-acyclic resolution of $\mathbb{L}_m$. In the following, we let $m = n+1$. Take a resolution $A^\bullet$ of $\mathbb{Z}/p^n$ by free $\mathbb{Z}/p^{n+1}$-modules

$$\ldots \to A^{-1} \to A^0 \to \mathbb{Z}/p^n \to 0.$$

We have that

$$\begin{aligned}
\mathbb{Z}/p^n \otimes^L_{\mathbb{Z}/p^{n+1}} K_{n+1}^\bullet &\cong A^\bullet \otimes_{\mathbb{Z}/p^{n+1}} \Gamma(U_{\overline{k}}, I'^\bullet) \\
&\cong \Gamma(U_{\overline{k}}, A^\bullet \otimes_{\mathbb{Z}/p^{n+1}} I'^\bullet) \\
&\cong R\Gamma(U_{\overline{k}}, \mathbb{Z}/p^n \otimes^L_{\mathbb{Z}/p^{n+1}} I'^\bullet) \\
&\cong R\Gamma(U_{\overline{k}}, \mathbb{L}_n) = K_n^\bullet
\end{aligned}$$

where the second isomorphism is due to that A^i are free $\mathbb{Z}/p^{n+1}$-modules, the third isomorphism is due to that $A^i \otimes I'^j$ is $R\Gamma$-acyclic and the last isomorphism is due to our assumption $\mathbb{L}_{n+1}/p^n \cong \mathbb{L}_n$. □

□

Proof of Theorem 7.14. This follows from the argument in [Sch13a, Theorem 8.4], for the sake of completeness let us repeat the argument in below.

First we claim that

$$H^i((X, X_{\bar{k}}), \mathbb{Z}/p^m) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}^a \cong H^i((X, X_{\bar{k}}), \mathbb{A}_{\mathrm{inf}}^a/p^m).$$

Indeed, when $m = 1$ this follows from Remark 5.10 (applied to $\mathbb{L} = \mathbb{F}_p$) and the general case follows from induction. Notice that the almost setting here is with respect to $[\hat{\mathfrak{m}}]$, the ideal generated by $([a], a \in \hat{\mathfrak{m}})$ where $\hat{\mathfrak{m}}$ is the maximal ideal in $\hat{\bar{k}}^\circ$. Now the sheaves $\mathbb{A}_{\mathrm{inf}}^a/p^m$ satisfy the hypotheses of the almost version of [Sch13a, Lemma 3.18]. Therefore we may pass to the inverse limit $\mathbb{A}_{\mathrm{inf}}^a$ and get an almost isomorphism

$$H^i((X, X_{\bar{k}}), \hat{\mathbb{Z}}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}^a \cong H^i((X, X_{\bar{k}}), \mathbb{A}_{\mathrm{inf}}^a).$$

Now we invert p and get almost isomorphisms

$$H^i((X, X_{\bar{k}}), \hat{\mathbb{Z}}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{inf}}^a \cong H^i((X, X_{\bar{k}}), \mathbb{B}_{\mathrm{inf}}^a).$$

Since $[\hat{\mathfrak{m}}]$ becomes the unit ideal in $B_{\mathrm{inf}}/\ker(\theta)$, multiplication by ξ^l (where ξ is any generator in $\ker(\theta)$) gives that

$$H^i((X, X_{\bar{k}}), \hat{\mathbb{Z}}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{inf}}/(\ker(\theta))^l \cong H^i((X, X_{\bar{k}}), \mathbb{B}_{\mathrm{inf}}/(\ker(\theta))^l).$$

Again the sheaves $\mathbb{B}_{\mathrm{inf}}/(\ker(\theta))^l$ satisfy the conditions in [Sch13a, Lemma 3.18], hence we have that

$$H^i((X, X_{\bar{k}}), \hat{\mathbb{Z}}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \cong H^i((X, X_{\bar{k}}), \mathbb{B}_{\mathrm{dR}}^+),$$

which is what we want by Remark 7.15. □

Finally let us show Theorem 1.3, which we restate below.

Theorem 7.18. *The Hodge–de Rham spectral sequence*

$$E_1^{j,i} = H^i(X, \Omega_X^j(\log D)) \Longrightarrow H^{i+j}(X, \Omega_X^\bullet(\log D))$$

degenerates, and there is a $\mathrm{Gal}(\bar{k}/k)$*-equivariant isomorphism*

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i(X, \Omega_X^\bullet(\log D)) \otimes_k B_{\mathrm{dR}}$$

preserving filtrations. In particular, there is also a $\mathrm{Gal}(\bar{k}/k)$*-equivariant isomorphism*

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \hat{\bar{k}} \cong \bigoplus_j H^{m-j}(X, \Omega_X^j(\log D)) \otimes_k \hat{\bar{k}}(-j).$$

Proof. By Theorem 7.14, we have

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \cong H^i((X, X_{\bar{k}}), \mathbb{B}_{\mathrm{dR}}^+).$$

In particular, $H^i((X, X_{\bar{k}}), \mathbb{B}_{\mathrm{dR}}^+)$ is a free B_{dR}^+-module of finite rank. This, together with Theorem 7.9, implies that

$$\sum_j \dim_k H^{i-j}(X, \Omega_X^j(\log D)) = \dim_{B_{\mathrm{dR}}}(H^i(X, \Omega_X^\bullet(\log D)) \otimes_k B_{\mathrm{dR}}),$$

hence the Hodge–de Rham spectral sequence degenerates. Also by Theorem 7.9, we get

$$H^i_{\text{ét}}(U_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H^i((X, X_{\bar{k}}), \mathbb{B}^+_{\text{dR}}) \otimes_{B^+_{\text{dR}}} B_{\text{dR}} \cong H^i(X, \Omega^\bullet_X(\log D)) \otimes_k B_{\text{dR}}.$$

□

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