ON RIGID VARIETIES WITH PROJECTIVE REDUCTION

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ABSTRACT. In this paper, we study smooth proper rigid varieties which admit formal models whose special fibers are projective. The main theorem asserts that the identity components of the associated rigid Picard varieties will automatically be proper. Consequently, we prove that p-adic Hopf varieties will never have a projective reduction. The proof of our main theorem uses the theory of moduli of semistable coherent sheaves.

CONTENTS

1. INTRODUCTION

In the famous series [BL93a], [BL93b], [BLR95a] and [BLR95b], Bosch, Lütkebohmert and Raynaud laid down the foundations relating formal and rigid geometry. The type of questions they treat in the series are mostly concerned with going from the rigid side to formal side. In this paper we will consider the opposite type of question, namely we will investigate to what extent properties on the formal side inform us about rigid geometry. More precisely, we will see what geometric consequences one can deduce under the assumption that the rigid space has a projective reduction.

Let K be a non-archimedean field with residue field k. Let X over K be a connected smooth proper rigid space with a K-rational point $x : \mathrm{Sp}(K) \to X$.

In this paper we prove the following:

Theorem 1.1. *Suppose that X has a formal model $\mathcal{X}$ whose special fiber $\mathcal{X}_0$ is projective over $\mathrm{Spec}(k)$, assume furthermore that Picard functor is represented by a quasi-separated rigid space. Then its identity component Pic_X^0 is proper.*

Note that if K is a p-adic field, then the properness of Pic^0_X implies Hodge symmetry of degree 1: $h^{1,0}(X) = h^{0,1}(X)$ (c.f. [HL17, Theorem 1.2]).

Let us make a historical remark on the representability of Picard functor in rigid geometry.

Remark 1.2.

(1) When K is discretely valued, Hartl and Lütkebohmert proved the representability of the Picard functor on the category of smooth rigid spaces over K under an additional assumption that X has a strict semistable formal model (c.f. [HL00]). They prove a structure theorem for the Picard space. In particular, the it is quasi-separated.
(2) In Warner's thesis it is proved that assuming K has characteristic 0, the Picard functor defined on a suitable category of adic spaces over K is represented by a separated rigid space over $\mathrm{Spa}(K, \mathcal{O})$ (c.f. [War17, Theorem 1.0.2]).
(3) In general, we expect the Picard functor of a proper smooth rigid space over K to be represented by a separated rigid space.

If X has a projective reduction then one can naturally define an open and closed sub-functor $\underline{\mathrm{Pic}}^P_{X/K}$ of $\underline{\mathrm{Pic}}_{X/K}$, see Definition 4.7.

Theorem 1.3 (Main Theorem). *Suppose that X has a formal model $\mathcal{X}$ whose special fiber $\mathcal{X}_0$ is projective over $\mathrm{Spec}(k)$. Then $\underline{\mathrm{Pic}}^P_{X/K}$ is a proper functor.*

The notion of a functor being proper is defined in Section 3. By Theorem 3.5, it suffices to prove $\underline{\mathrm{Pic}}^P_{X/K}$ is bounded. We use moduli of semistable sheaves to show this.

Let us summarize how this paper is organized. In Section 4 we make an observation that there is a well defined specialization from the K-group of coherent sheaves on X to that of $\mathcal{X}_0$. Therefore a chosen ample invertible sheaf on $\mathcal{X}_0$ will enable us to associate Hilbert polynomials. As a byproduct, we find that non-archimedean Hopf surfaces do not have any formal model with projective reduction, see Proposition **??**.

From Section 5 on, we fix a chosen ample invertible sheaf on $\mathcal{X}_0$. Then we define (semi)-stability of coherent sheaves on X and generalize a result of Langton, namely we prove that a semistable coherent sheaf F on X always has a formal model $\mathcal{F}$ such that its reduction $\mathcal{F}_0$ is semistable, which justifies the title of this section. Consequently, any line bundle on X has a formal model whose reduction is semistable.

In the last section we construct an auxiliary quasi-compact space W which "surjects" onto $\underline{\mathrm{Pic}}^P_X$; this completes the proof of our main theorem.

2. NOTATIONS

Let K be a non-archimedean field with value group $\Gamma \subset \mathbb{R}$, ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{m}$ and residue field k. Throughout this paper, $\mathcal{X}$ will be a proper admissible formal model of a smooth proper connected rigid space $X = \mathcal{X}^{\mathrm{rig}}$ over $\mathcal{O}$. For simplicity we will assume X has a K-rational point $x : \mathrm{Sp}(K) \to X$. We use almost the same notation as in [BL93a] except that we use $(\cdot)^{\mathrm{rig}}$ to denote the generic fiber of an admissible formal scheme. We use roman letters to denote rigid objects and curly letters to denote formal objects. We also denote the level π (resp.

γ), namely modulo π (resp. modulo π_γ for some $|\pi_\gamma| = \gamma$), of a formal objects by subscript π (resp. γ).

3. Preliminaries on Picard functor

In this section, we will have a general discussion of (Picard) functor in the context of rigid spaces and adic spaces.

Following [War17], we consider functors defined on a suitable subcategory of adic spaces over K. Let us denote $(\mathbf{V}/K)$ the full subcategory of adic spaces over $\mathrm{Spa}(K, \mathcal{O})$, its objects are adic spaces locally of finite type over $\mathrm{Spa}(K', R') \to \mathrm{Spa}(K, \mathcal{O})$ where K'/K is a non-archimedean field extension, R' gives a (not necessarily rank 1) valuation on K' with $R' \cap K = \mathcal{O}$. In the following we will use underlined notation to denote a functor, and the representing space (if exists) will be denoted without underline. Note that this is opposite to the convention of Grothendieck but agrees with that in loc. cit.

In this paper, we consider the Picard functor

$$\underline{\mathrm{Pic}}_{X/K} \colon (\mathbf{V}/K)^{op} \to \mathcal{S}ets$$

defined by

$$\underline{\mathrm{Pic}}_{X/K}(V) = \left\{ \begin{array}{l} \mathcal{L} \text{ a line bundle on } X^{\mathrm{ad}} \times_{\mathrm{Spa}(K, \mathcal{O})} V, \\ \lambda \colon \mathcal{O}_V \xrightarrow{\sim} (x, id)^* \mathcal{L} \text{ an isomorphism} \end{array} \right\} / \mathrm{Isom}.$$

Note that this is a special case of the Picard functor defined in [War17]. Let us make the following definition.

Definition 3.1. Let $F \colon (\mathbf{V}/K)^{op} \to \mathcal{S}ets$ be a functor.

(1) F is said to be *partially proper* if any element in $F(\mathrm{Spa}(\mathrm{Frac}(R), R))$ can be extended to a unique element in $F(\mathrm{Spa}(\mathrm{Frac}(R), R^+))$. Here R is a rank 1 valuation ring over $\mathcal{O}$ and R^+ is a valuation subring of R with the same fraction field.

(2) F is said to be *bounded* if there exists a quasi-compact rigid space W over K and an element $\xi \in F(W)$ such that for any $\eta \in F(\bar{K})$, there exists $P \in W(\bar{K})$ with $\eta \cong P^*(\xi)$.

(3) F is said to be *proper* if it is partially proper and bounded.

In the definition of boundedness of functor, it is equivalent if we ask the space W to be an affinoid space. This definition is justified by Proposition 3.2 below.

Proposition 3.2. *Let $F \colon (\mathbf{V}/K)^{op} \to \mathcal{S}ets$ be a (partially) proper functor. If F is represented by a quasi-separated rigid space Y over $\mathrm{Sp}(K)$. Then Y is (partially) proper.*

Huber has defined the notion of partial properness (c.f. [Hub96, Definition 1.3.3]) for analytic adic spaces. He showed a valuative criterion of partial properness for maps between analytic adic spaces (c.f. [Hub96, Corollary 1.3.9]). Combined with the work in [Lüt90], Huber was able to show that a map of rigid spaces over discretely valued field is (partially) proper if and only if the map of corresponding analytic adic spaces is (partially) proper (c.f. [Hub96, Remark 1.3.19]). Temkin generalized the result of Lütkebohmert, c.f. [Tem00, Theorem 4.1]. In particular, we now know that a quasi-separated map of rigid spaces over an arbitrary non-archimedean field is partially proper if and only if the map of corresponding analytic adic spaces satisfies the valuative criterion in [Hub96, Lemma 1.3.10].

Proof. By definition of F being partially proper, we see that Y satisfies the valuative. Since Y is quasi-separated, it is partially proper by [Hub96, Corollary 1.3.9].

If F is proper, then by the above we know that Y is partially proper. It suffices to show that F being bounded implies that Y is quasi-compact. This follows from the Lemma below. $\square$

Lemma 3.3. *Let $f : W \to Y$ be a morphism of rigid spaces. Suppose W is quasi-compact, Y is quasi-separated and f is surjective on classical Tate points. Then Y is also quasi-compact.*

Proof. Let $\{U_i\}_{i \in I}$ be an affinoid admissible covering of Y. Because W is quasi-compact and f is surjective we see that finitely many of the U_i's cover Y set-theoretically. Because Y is quasi-separated we see that these U_i's form an admissible covering of Y. $\square$

It is worth noticing the following (counter)-example if we drop the assumption of Y being quasi-separated.

Example 3.4. Let $K = \mathbb{C}_p$ and let W be the closed disc of radius 1. Choose an irrational number r with $0 < r < 1$. Let Y be the gluing of infinitely many closed discs of radius 1, where we glue all the radius greater than r parts together and do the same to all the radius less than r parts. There is an obvious map from W to Y that is a surjection on classical Tate points. It is easy to see that Y is not quasi-compact.

It is a consequence of [KL16, Theorem 2.3.3] that $\underline{\mathrm{Pic}}_{X/K}$ is always partially proper, see Theorem 3.5 below.

Theorem 3.5. $\underline{\mathrm{Pic}}_{X X/K}$ *is partially proper.*

Proof. Let R and R^+ be as in Definition 3.1. By [KL16, Theorem 2.3.3] we see that the restriction functor from the category of coherent sheaves on $X^{\mathrm{adic}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(\mathrm{Frac}(R), R^+)$ to the category of coherent sheaves on $X^{\mathrm{adic}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(\mathrm{Frac}(R), R)$ is an equivalence and preserves finite locally free coherent sheaves. Let $\mathcal{L}_R$ on $X^{\mathrm{adic}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(\mathrm{Frac}(R), R)$ be a rigidified line bundle. By the equivalence of categories we see that $\mathcal{L}_R$ extends uniquely as a rigidified line bundle $\mathcal{L}_{R^+}$ on $X^{\mathrm{adic}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(\mathrm{Frac}(R), R^+)$.

The argument above shows that the Picard functor satisfies the valuative criterion, namely any map

$$f : \mathrm{Spa}(\mathrm{Frac}(R), R) \to \underline{\mathrm{Pic}}_{X/K}$$

always extends to a map

$$f : \mathrm{Spa}(\mathrm{Frac}(R), R^+) \to \underline{\mathrm{Pic}}_{X/K}$$

which is what we need to show. $\square$

4. Specialization of K group

Suppose we have a class in $K_0(X)$ represented by a coherent sheaf F. Then we can find some formal model $\mathcal{F}$ of F, by which we mean an $\mathcal{O}$-torsion free finitely presented $\mathcal{O}_{\mathcal{X}}$-module with generic fibre isomorphic to F. After reduction we get a coherent sheaf $\mathcal{F}_0$ on $\mathcal{X}_0$. Different formal models of F will differ by an $\mathcal{O}$-torsion finitely presented sheaf on $\mathcal{X}$. Let us prove a lemma on the $\mathcal{O}$-module structure of such sheaves which we believe is interesting on its own.

Lemma 4.1. *Let $\mathcal{A}$ be a topologically finitely presented $\mathcal{O}$-algebra and let M be an $\mathcal{O}$-torsion finitely presented $\mathcal{A}$-module. Then as an $\mathcal{O}$-module, we have the following decomposition:*

$$M = \bigoplus_{i=1}^{k} (\mathcal{O}/\pi_i)^{\oplus \lambda_i}$$

for some (finite or countable) cardinals $\lambda_1, \ldots, \lambda_k$.

Proof. Without loss of generality, we may assume $\mathcal{A} = (\mathcal{O}/\pi)[x_1, \ldots, x_n]/I$ with I finitely generated. We will do induction on the dimension of the support of M, which we denote as d. Now without loss of generality we may assume the dimension of $|\operatorname{Spec}(\mathcal{A})| = |\operatorname{Spec}(\mathcal{A} \otimes_{\mathcal{O}} k)|$ is d. By Noether normalization, we may find a finite morphism $k[x_1, \ldots, x_d] \to \mathcal{A} \otimes_{\mathcal{O}} k$. Lifting the images of x_i's gives us a morphism $\mathcal{B} = \mathcal{O}[x_1, \ldots, x_d] \to \mathcal{A}$ which is universally closed and of finite presentation as an algebra, hence of finite presentation as a module. Now regarding M as a finitely presented $\mathcal{B}$-module, we reduce to the situation where $\mathcal{A} = \mathcal{B}$. Localizing at the generic point of the special fiber $\mathfrak{p} = \mathfrak{m}[x_1, \ldots, x_d]$, we see that $M_{\mathfrak{p}}$ is a finitely presented $\mathcal{B}_{\mathfrak{p}}$-module. Because $\mathcal{B}_{\mathfrak{p}}$ is a valuation ring which is unramified over $\mathcal{O}$, i.e., they have the same value group. We see that $M_{\mathfrak{p}} = \bigoplus_{i=1}^{k} (\mathcal{B}_{\mathfrak{p}}/\pi_i \mathcal{B}_{\mathfrak{p}})$. By clearing denominator, this gives rise to a morphism $\bigoplus_{i=1}^{k} \mathcal{B} \to M$, and hence an injection $\phi : \bigoplus_{i=1}^{k} (\mathcal{B}/\pi_i) \hookrightarrow M$ after multiplying some element $f \notin \mathfrak{p}$.

Next, we claim that ϕ is universally injective as an $\mathcal{O}$-module and the quotient is a finitely presented $\mathcal{B}$-module whose support has dimension less than d. The claim on the dimension of the support is easy and follows from the fact that this map is an isomorphism on the generic point of the special fiber $\mathfrak{p}$. To see why this map is universally injective as an $\mathcal{O}$-module, let us consider the following commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{i=1}^{k}(\mathcal{B}/\pi_i) & \lhook\joinrel\longrightarrow & M \\
\big\downarrow & & \big\downarrow \\
\bigoplus_{i=1}^{k}(\mathcal{B}_{\mathfrak{p}}/\pi_i \mathcal{B}_{\mathfrak{p}}) & \overset{\sim}{\longrightarrow} & M_{\mathfrak{p}}
\end{array}
\quad .
$$

Now by [Sta17, Lemma Tag 05CI] and [Sta17, Lemma Tag 05CJ], it suffices to show that $\mathcal{B}/\pi_i \to \mathcal{B}_{\mathfrak{p}}/\pi_i \mathcal{B}_{\mathfrak{p}}$ is universally injective as an $\mathcal{O}$-module. This in turn would follow from the fact that $\mathcal{B} \to \mathcal{B}_{\mathfrak{p}}$ is universally injective as an $\mathcal{O}$-module, by [Sta17, Theorem Tag 058K] it suffices to show $\mathcal{B}/\pi_i \to \mathcal{B}_{\mathfrak{p}}/\pi_i \mathcal{B}_{\mathfrak{p}}$ is injective as an $\mathcal{O}$-module which one verifies directly.

Consider the short exact sequence $0 \to \bigoplus_{i=1}^{k}(\mathcal{B}/\pi_i) \to M \to Q \to 0$. It is easy to see that $\bigoplus_{i=1}^{k}(\mathcal{B}/\pi_i)$ has the form we want. We see that this sequence is universally exact with respect to $\mathcal{O}$-module structure and Q is a finitely presented $\mathcal{B}/(f)$-module for some $f \notin \mathfrak{m}[x_1, \ldots, x_d]$ and therefore is of the form $Q = \bigoplus_{i=1}^{k'} (\mathcal{O}/\pi_i')^{\oplus \lambda_i'}$ by the induction hypothesis. Now by [Sta17, Theorem Tag 058K], we see that the short exact sequence above splits. Therefore we see that M, as an $\mathcal{O}$-module, has the form we want. $\qquad \square$

Now we can state and prove the key observation of this paper.

Theorem 4.2. *The association $[F] \mapsto [\mathcal{F}_0]$ gives rise to a well-defined map from $K_0(X)$ to $K_0(\mathcal{X}_0)$.*

Proof. The first thing we have to verify is that for two different formal models of F, say $\mathcal{F}^1$ and $\mathcal{F}^2$, we will have $[\mathcal{F}_0^1] = [\mathcal{F}_0^2]$ as classes in $K_0(\mathcal{X}_0)$. Viewing $\mathcal{F}^1$ and $\mathcal{F}^2$ as two lattices inside F, then after multiplying $\mathcal{F}^1$ by an element in $\mathfrak{m}$ with sufficiently large valuation, we may assume it is a subsheaf of $\mathcal{F}^2$.

In the situation above, we have a short exact sequence

$$0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to Q \to 0$$

Applying $\otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}/\mathfrak{m}\mathcal{O}_\mathcal{X}$ we get

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}/\mathfrak{m}\mathcal{O}_\mathcal{X}, Q) \to \mathcal{F}_0^1 \to \mathcal{F}_0^2 \to Q_0 \to 0.$$

Observe that $\mathrm{Tor}^1_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}/\mathfrak{m}\mathcal{O}_\mathcal{X}, Q) = \ker(\mathfrak{m} \otimes Q \to Q)$, which we shall denote as Q_0'.

Now we only have to show $[Q_0'] = [Q_0]$. We will define canonical filtrations on both coherent sheaves and use Lemma 4.1 to see their successive quotients are naturally isomorphic. For every $\gamma \in \Gamma$, we define

$$\mathrm{Fil}^\gamma(Q_0) = \mathrm{Im}(\ker(\cdot\pi_\gamma : Q \to Q) \to Q_0)$$

and

$$\mathrm{Fil}^\gamma(Q_0') = (\pi_\gamma \otimes Q) \cap \ker(\mathfrak{m} \otimes Q \to Q)$$

where π_γ is any element in $\mathcal{O}$ with valuation γ. Since Q is an $\mathcal{O}$-torsion finitely presented $\mathcal{O}_\mathcal{X}$-module, we see that as γ goes from 0 to some $|\pi|$ the first (resp. second) filtration is an increasing (resp. decreasing) filtrations of coherent subsheaves and both filtrations are exhaustive and separated. We claim that both filtrations only change at finitely many γ_i and there are natural isomorphisms between corresponding successive quotients. Because both claims are local properties, after choosing a finite covering of $\mathcal{X}$ by affine formal schemes we may reduce to the situation where $\mathcal{X} = \mathrm{Spf}(\mathcal{A})$. Now we have $Q = \widetilde{M}$ for some $\mathcal{O}$-torsion finitely presented $\mathcal{A}$-module M. By Lemma 4.1 we see that as an $\mathcal{O}$-module,

$$M = \bigoplus_{i=1}^{k} (\mathcal{O}/\pi_i)^{\oplus \lambda_i}.$$

We may assume $|\pi_1| < \cdots < |\pi_k|$. Now by direct computation, we see that both filtrations only change when $\gamma = |\pi_i|$ for $1 \le i \le k$. On the successive quotient we may define

$$f_i : \mathrm{Gr}^i(Q_0) = \frac{\mathrm{Im}(\ker(\cdot\pi_i : Q \to Q) \to Q_0)}{\mathrm{Im}(\ker(\cdot\pi_{i+1} : Q \to Q) \to Q_0)} \to \mathrm{Gr}^i(Q_0') = \frac{(\pi_i \otimes Q) \cap \ker(\mathfrak{m} \otimes Q \to Q)}{(\pi_{i-1} \otimes Q) \cap \ker(\mathfrak{m} \otimes Q \to Q)}$$

by $f_i(x) = \overline{\pi_i \tilde{x}}$ where $\tilde{x}$ is any lift of x in $\mathrm{Im}(\ker(\cdot\pi_i : Q \to Q) \to Q_0)$. Again one checks directly that f_i is a well-defined isomorphism and commutes with localization. This proves our claim that $[Q_0'] = [Q_0]$, which implies that the class $[\mathcal{F}_0]$ is well defined.

Secondly we note that every short exact sequence of coherent sheaves on X will have a formal model, i.e., a short exact sequence of $\mathcal{O}$-flat finitely presented sheaves on $\mathcal{X}$. Indeed, consider $0 \to F \to G \to H \to 0$ a short exact sequence of coherent sheaves on X. We choose a formal model of G, say $\mathcal{G}$. Then $\mathcal{F} := F \cap \mathcal{G}$ is a saturated finitely presented (c.f. [BL93a, Proposition 1.1 and Lemma 1.2]) sheaf whose generic fiber is F. Hence $\mathcal{H} := \mathcal{G}/\mathcal{F}$ is a $\mathcal{O}$-flat finitely presented sheaf on $\mathcal{X}$ whose generic fiber is H. Now because $\mathcal{H}$ is $\mathcal{O}$-flat, after tensoring with k we get a

short exact sequence $0 \to \mathcal{F}_0 \to \mathcal{G}_0 \to \mathcal{H}_0 \to 0$. So we see this definition respects the relation coming from short exact sequences. This proves our theorem. $\square$

Remark 4.3. From the proof of Theorem 4.2 we see that for any two choices of formal models $\mathcal{F}^i$ of F, we can define a decreasing filtration $\mathrm{Fil}^i(\mathcal{F}_0^1)$ and an increasing filtration $\mathrm{Fil}^i(\mathcal{F}_0^2)$ along with maps $\phi^i : \mathrm{Fil}^i(\mathcal{F}_0^1) \to (\mathcal{F}_0^2)/(\mathrm{Fil}^i(\mathcal{F}_0^2))$ such that these maps give rise to isomorphisms between $\mathrm{Gr}^i(\mathcal{F}_0^1)$ and $\mathrm{Gr}^i(\mathcal{F}_0^2)$. This will be used later, see Theorem 5.9.

From now on we will assume that $\mathcal{X}_0$ is projective, with a fixed ample invertible sheaf H. Consider $\mathrm{Spa}(K', R') \to \mathrm{Spa}(K, \mathcal{O})$ as in the definition of the category $\mathbf{V}/K$. We may form a formal model of $X_{K'}$ by taking $\mathcal{X} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spf}\,\mathcal{O}_{K'}$. The special fibre of this formal model will be $\mathcal{X}_0 \times_k k'$ and carry the pullback of H which is still ample, where k' is the residue field of $\mathcal{O}_{K'}$. Then we can define the Hilbert polynomial of a coherent sheaf on X (or more generally $X^{\mathrm{ad}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(K', R')$) in the following way:

Definition 4.4. For any coherent sheaf F on X, we define the *Hilbert polynomial* of F to be

$$P_H(F)(n) = \chi_{\mathcal{X}_0}(Sp(F) \otimes H^{\otimes n})$$

where Sp is the specialization map of K groups from previous theorem.

More generally for any coherent sheaf F on $X^{\mathrm{ad}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(K', R')$, we define its Hilbert polynomial to be that of its restriction to $X^{\mathrm{ad}} \times_{\mathrm{Spa}(K,\mathcal{O})} \mathrm{Spa}(K', \mathcal{O}_{K'})$ with respect to the formal model $\mathcal{X}_0 \times_k k'$ and the pullback of H.

Proposition 4.5. *Suppose $F \in Coh(X \times S)$ is a flat family of coherent sheaves on X parametrized by a quasi-compact quasi-separated connected rigid space S, and suppose $\mathcal{X}_0$ is projective with an ample invertible sheaf H. Then any two fibres of F will have the same Hilbert polynomial (i.e., the Hilbert polynomial is locally constant in flat families).*

More generally if $F \in Coh(X^{\mathrm{ad}} \times_{\mathrm{Spa}(K,\mathcal{O})} V)$ is a flat family of coherent sheaves with $(V \to \mathrm{Spa}(K', R')) \in \mathbf{V}/K$. Then the Hilbert polynomial function gives a partition of V into union of connected components.

Proof. Let $\mathcal{S}$ be a formal model of S. Then $\mathcal{X} \times \mathcal{S}$ is a formal model of $X \times S$, and F extends to a coherent sheaf $\mathcal{F}$ on $\mathcal{X} \times \mathcal{S}$.

Applying [BL93b, Theorem 4.1], we see that after a possible admissible formal blowing-up of $\mathcal{S}$ we may assume that $\mathcal{F}$ is flat over $\mathcal{S}$. Hence $\mathcal{F}_0$ will be flat over $\mathcal{S}_0$. Then it follows from cohomology and base change (c.f. [Mum08, Corollary 1 in Section 5]) that the reduction of two special fibers of F have the same Hilbert polynomial.

We see that the Hilbert polynomial is locally constant in a flat family, so the statement above would still hold even if we do not assume quasi-compactness or quasi-separatedness of S.

To prove the more general statement it suffices to notice that for a locally finite type adic space V over $\mathrm{Spa}(K', R')$, its connected components are in bijection with the connected components of $V \times_{\mathrm{Spa}(K',R')} \mathrm{Spa}(K', \mathcal{O}_{K'})$. Now by locally constancy in the case of rigid space proved above, we see that the Hilbert polynomial gives rise to a partition of $V \times_{\mathrm{Spa}(K',R')} \mathrm{Spa}(K', \mathcal{O}_{K'})$ into union of connected components. $\square$

The above discussion already yields the following interesting consequence for the geometry of non-archimedean Hopf surfaces, which were defined and studied by Voskuil in [Vos91].

Proposition 4.6. *Non-archimedean Hopf surfaces over a non-archimedean field have no projective reduction.*

Proof. In the paper mentioned above, Voskuil proved that there is a flat family of line bundles on a Hopf surface parametrized by $\mathbb{G}_m$ with identity in $\mathcal{G}_m$ corresponds to $\mathcal{O}_X$. Moreover, he proved that there are infinitely many nontrivial (i.e., not isomorphic to $\mathcal{O}_X$) line bundles in that family which possess sections.

Now suppose that a Hopf surface X has a projective reduction. Then we can define Hilbert polynomials as above. Because $\mathbb{G}_m$ is connected, the line bundles it parametrizes will have the same Hilbert polynomial by Proposition 4.5. Let $\mathcal{L}$ be a nontrivial line bundle possessing a nontrivial section, so that we get an injection $0 \to \mathcal{O}_X \to \mathcal{L} \to Q \to 0$. We observe immediately that the cokernel Q has zero Hilbert polynomial, which means Q is a zero sheaf. This is a contradiction as $\mathcal{L}$ is assumed to be a nontrivial line bundle. $\square$

Using Hilbert polynomial we may define a natural partition on any moduli functor of coherent sheaves, as illustrated below.

Definition 4.7. Assume X has a projective reduction and fix a projective reduction $\mathcal{X}_0$ along with an ample line bundle on it. Let $\underline{\mathrm{Pic}}^P_{X/K}$ be the open and closed sub-functor of $\underline{\mathrm{Pic}}_{X/K}$ parametrizing line bundles having Hilbert polynomial the same as that of $\mathcal{O}_X$.

Remark 4.8.
  (1) This is an open and closed sub-functor because of Proposition 4.5.
  (2) It is not clear to the author if the definition above depends on choices of the projective reduction and the ample line bundle on it. The original purpose of this paper was on the behavior of Pic^0_X, regarding this direction we do not have to care about this issue.

5. Semistable Reduction of Coherent Sheaves

In this section we prove a semistable reduction type theorem for semistable coherent sheaves, following a similar method as in [Lan75]. Following [HL10], we make the definitions below. From now on we will assume all the formal schemes appearing below have projective special fibers.

Definition 5.1. Let $\mathcal{F}$ be an $\mathcal{O}$-flat finitely presented sheaf on $\mathcal{X}$.
  (1) The *Hilbert polynomial* of $\mathcal{F}$ is the Hilbert polynomial of $\mathcal{F}_0$.
  (2) The *dimension* of $\mathcal{F}$ is the dimension of its support, denoted as $\dim \mathcal{F}$.
  (3) The *rank* of $\mathcal{F}$ is the leading coefficient of its Hilbert polynomial divided by that of $\mathcal{O}_X$, denoted as $\mathrm{rk}(\mathcal{F})$.
  (4) $\mathcal{F}$ is said to be *pure* if any coherent subsheaf of $\mathcal{F}$ with a flat quotient has dimension $n = \dim(\mathcal{F})$.
  (5) The *dimension* of a coherent sheaf F on X is the dimension of its support. It is called *pure* if any nonzero coherent subsheaf has the same dimension.

The following lemma describes the relation between pureness of a coherent sheaf and pureness of its formal model.

Lemma 5.2. *Let F be a coherent sheaf on X. The following are equivalent:*

(1) *F is pure;*
(2) *There is a formal model of F which is pure;*
(3) *Any formal model of F is pure.*

Proof. This lemma follows immediately from the fact that the category of coherent subsheaves in F with morphisms being inclusions and the category of finitely presented subsheaves in $\mathcal{F}$ with $\mathcal{O}$-flat quotient and morphisms being inclusions are equivalent. With one direction functor being intersecting with $\mathcal{F}$ and the functor of taking generic fiber in the reverse direction. $\square$

Definition 5.3. Let $\mathcal{F}$ be an $\mathcal{O}$-flat finitely presented sheaf on $\mathcal{X}$.

(1) $\mathcal{F}$ is said to be *semistable* if it is pure and for any finitely presented subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $p(\mathcal{G}) \leq p(\mathcal{F})$, where $p(\mathcal{F})$ is the reduced Hilbert polynomial, i.e. Hilbert polynomial divided by its leading coefficient, of a finitely presented sheaf $\mathcal{F}$. We denote the $(k+1)$-th coefficient of reduced Hilbert polynomial of $\mathcal{F}$ by $a_k(\mathcal{F})$.
(2) Let $f = x^n + b_1 x^{n-1} + \cdots + b_n$ be the reduced Hilbert polynomial of the maximal destabilizing sheaf of $\mathcal{F}_0$. We define the *maximal destabilizing sheaf of codimension k* of $\mathcal{F}_0$ as the maximal subsheaf $B \subset \mathcal{F}_0$ such that $a_i(B) = b_i$ for all $1 \leq i \leq k$. The existence of the maximal destabilizing sheaf of codimension k is guaranteed by Harder–Narasimhan theory.
(3) $\mathcal{F}$ is *semistable of codimension k* if for any coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{O}$-flat quotient, we have $a_i(\mathcal{G}) \leq a_i(\mathcal{F})$ for all $1 \leq i \leq k$.
(4) The *semistable codimension* of $\mathcal{F}$ is the biggest k for which $\mathcal{F}$ is semistable of codimension k.
(5) We make similar definitions for a coherent sheaf F on X.

Remark 5.4. One word on the existence of Harder–Narasimhan filtration in the rigid setup is necessary. One can check the proof of existence and uniqueness of Harder–Narasimhan filtrations in the algebraic setup (c.f. [HL10, Theorem 1.3.4]) to see that we only need Noetherianness of coherent sheaves and additivity of Hilbert polynomial (with respect to short exact sequences) to do that. In the rigid setup both properties still hold.

Lemma 5.5. *Let F be a coherent sheaf on X. The following are equivalent:*

(1) *F is semistable of codimension k;*
(2) *There is a formal model of F which is semistable of codimension k;*
(3) *Any formal model of F is semistable of codimension k.*

Proof. The proof is the same as that of the previous lemma. $\square$

Definition 5.6. Let $\mathcal{F}$ be an $\mathcal{O}$-flat finitely presented sheaf on $\mathcal{X}$. Consider an exact sequence $0 \to \mathcal{B}_0 \to \mathcal{F}_0 \to \mathcal{G}_0 \to 0$ of coherent sheaves on $\mathcal{X}_0$. We say such a sequence is *liftable modulo $\pi \in \mathfrak{m}$* if there is an exact sequence of finitely presented sheaves $0 \to \mathcal{B} \to \mathcal{F} \otimes \mathcal{O}/(\pi) \to \mathcal{G} \to 0$ whose special fiber is the given sequence above with $\mathcal{G}$ flat over $\mathcal{O}/(\pi)$.

Lemma 5.7. *Let $\mathcal{F}$ be an $\mathcal{O}$-flat finitely presented sheaf on $\mathcal{X}$. Consider $0 \to \mathcal{B}_0 \to \mathcal{F}_0 \to \mathcal{G}_0 \to 0$, a short exact sequence of coherent sheaves on $\mathcal{X}_0$.*

(1) *If* $\mathrm{Hom}(\mathcal{B}_0, \mathcal{G}_0) = \{0\}$, *there exists a biggest (in the sense of having maximal valuation)* $\pi \in \mathfrak{m}$ *such that the sequence is liftable modulo* π. *Here* π *could be* 0, *by which we mean the sequence can be lifted all the way to* $\mathcal{O}$.

(2) *In the following, assume the* π *above is not* 0. *Let* $\mathcal{F}^{(1)}$ *be the kernel of the composition* $\mathcal{F} \to \mathcal{F} \otimes \mathcal{O}/(\pi) \to \mathcal{G}$. *Then* $\mathcal{F}^{(1)}$ *is finitely presented, and we have a short exact sequence* $0 \to \mathcal{G} \to \mathcal{F}^{(1)} \otimes \mathcal{O}/(\pi) \to \mathcal{B} \to 0$. *Furthermore,* $\mathcal{F}/\mathcal{F}^{(1)}$ *is* π-*torsion.*

(3) *Moreover the reduction of our short exact sequence above* $0 \to \mathcal{G}_0 \to \mathcal{F}_0^{(1)} \to \mathcal{B}_0 \to 0$ *does not split.*

Proof. Proof of (1). Consider the formal Quot scheme $\mathcal{Q} = \mathrm{Quot}_{\mathcal{F}/\mathcal{X}/\mathcal{O}}$. Our condition says the point $P \in \mathcal{Q}_0(k)$ corresponding to $\mathcal{G}_0$ is an isolated point. So the component corresponding to P is of the form $\mathcal{Q}_p = \mathrm{Spec}(\mathcal{O}/(\pi))$ which means exactly that our sequence is liftable modulo π but not modulo an element with bigger valuation.

Proof of (2). We consider $\pi\mathcal{F}^{(1)} \subset \pi\mathcal{F} \subset \mathcal{F}^{(1)}$, whose corresponding quotients give us a short exact sequence

$$0 \to \mathcal{G} = \pi\mathcal{F}/\pi\mathcal{F}^{(1)} \to \mathcal{F}^{(1)} \otimes \mathcal{O}/(\pi) = \mathcal{F}^{(1)}/\pi\mathcal{F}^{(1)} \to \mathcal{B} = \mathcal{F}^{(1)}/\pi\mathcal{F} \to 0$$

which is what we want, and it is immediate that $\mathcal{F}/\mathcal{F}^{(1)}$ is π-torsion.

Proof of (3). Suppose on the contrary that the sequence $0 \to \mathcal{G}_0 \to \mathcal{F}_0^{(1)} \to \mathcal{B}_0 \to 0$ splits and view $\mathcal{B}_0$ as a subsheaf in $\mathcal{F}_0^{(1)}$. By a limit argument we will have $0 \to \mathcal{B}' \to \mathcal{F}^{(1)}/\pi'\mathcal{F}^{(1)} \to \mathcal{G}' \to 0$ for some $\pi' \in \mathfrak{m}$ where $\mathcal{G}'$ is flat over $\mathcal{O}/\pi'$, and we may choose π' so that the composite $\pi\mathcal{F} \hookrightarrow \mathcal{F}^{(1)} \twoheadrightarrow \mathcal{G}'$ is surjective. Then we consider $\tilde{\mathcal{B}}$ which is defined by the following exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{B}} & \longrightarrow & \mathcal{F}^{(1)}/\pi\pi'\mathcal{F} & \longrightarrow & \mathcal{G}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{F}^{(1)}/\pi'\mathcal{F}^{(1)} & \longrightarrow & \mathcal{G}' & \longrightarrow & 0
\end{array}
$$

We claim that the exact sequence

$$0 \to \tilde{\mathcal{B}} \to \mathcal{F}/\pi\pi'\mathcal{F} \to \tilde{\mathcal{G}} \to 0$$

is flat over $\mathcal{O}/\pi\pi'$, which would contradict the maximality of (the valuation of) π. Indeed, by considering $\tilde{\mathcal{B}} \subset \mathcal{F}^{(1)}/\pi\pi'\mathcal{F} \subset \mathcal{F}/\pi\pi'\mathcal{F}$, we get $0 \to \mathcal{G}' \to \tilde{\mathcal{G}} \to \mathcal{G} \to 0$. To prove $\tilde{\mathcal{G}}$ is flat over $\mathcal{O}/\pi\pi'$ it suffices to show that the map $\tilde{\mathcal{G}} \to \mathcal{G}$ is the same as tensoring $\mathcal{O}/\pi$, which in turn is equivalent to saying $\tilde{\mathcal{B}} + \pi\mathcal{F}/\pi\pi'\mathcal{F} = \mathcal{F}^{(1)}/\pi\pi'\mathcal{F}$, and this equality follows from the fact that the composite $\pi\mathcal{F} \hookrightarrow \mathcal{F}^{(1)} \to \mathcal{G}'$ is surjective. $\square$

The key result of this section is the following theorem which generalizes Langton's result in [Lan75].

Theorem 5.8. *Let* X *be a rigid space with a formal model* $\mathcal{X}$. *Assume* $\mathcal{X}_0$ *is projective with a fixed ample invertible sheaf* H.

(1) *Let* F *be a pure coherent sheaf on* X. *Then there exists a formal model* $\mathcal{F}$ *on* $\mathcal{X}$ *with reduction* $\mathcal{F}_0$ *pure on* $\mathcal{X}_0$. *Conversely, if* F *admits a formal model* $\mathcal{F}$ *with pure reduction, then* F *itself is pure.*

(2) *Let F be a semistable coherent sheaf on X. Then there exists a formal model $\mathcal{F}$ on $\mathcal{X}$ with reduction $\mathcal{F}_0$ semistable on $\mathcal{X}_0$. Conversely, if F admits a formal model $\mathcal{F}$ with semistable reduction, then F itself is semistable.*

Proof. Proof of (1). We choose an arbitrary formal model $\mathcal{F}$ on $\mathcal{X}$; by Lemma 5.2 we know $\mathcal{F}$ is pure. Now we want to find a finitely presented subsheaf $\mathcal{F}' \subset \mathcal{F}$ with torsion quotient and special fiber $\mathcal{F}'_0$ pure on $\mathcal{X}_0$.

Let $\mathcal{B}_0 \subset \mathcal{F}_0$ be the maximal coherent subsheaf whose support is not of dimension $\dim(F)$ (from now on we will call it the maximal torsion subsheaf of $\mathcal{F}_0$) and denote the quotient by $\mathcal{G}_0$. It is easy to see that $\mathrm{Hom}(\mathcal{B}_0, \mathcal{G}_0) = 0$, so applying Lemma 5.7 to the short exact sequence $0 \to \mathcal{B}_0 \to \mathcal{F}_0 \to \mathcal{G}_0 \to 0$ gives a finitely presented subsheaf $\mathcal{F}^{(1)} \subset \mathcal{F}$ and a short exact sequence $0 \to \mathcal{G}_0 \to \mathcal{F}_0^{(1)} \to \mathcal{B}_0 \to 0$. After repeatedly applying the above procedure, we will obtain a sequence of finitely presented sheaves

$$\mathcal{F} = \mathcal{F}^{(0)} \supset \mathcal{F}^{(1)} \supset \cdots .$$

Note that all of the $\mathcal{F}_0^{(i)}$'s have the same Hilbert polynomial. We get short exact sequences

$$0 \to \mathcal{B}_0^{(i)} \to \mathcal{F}_0^{(i)} \to \mathcal{G}_0^{(i)} \to 0$$

and

$$0 \to \mathcal{G}_0^{(i)} \to \mathcal{F}_0^{(i+1)} \to \mathcal{B}_0^{(i)} \to 0.$$

We claim that after finitely many steps the maximal torsion subsheaf of $\mathcal{F}_0^{(i)}$, denoted by $\mathcal{B}_0^{(i)}$, will have either dimension or rank smaller than that of $\mathcal{B}_0$. According to this claim, after N steps, $\mathcal{B}_0^{(N)}$ will be 0. This implies $\mathcal{F}_0^{(N)}$ contains no maximal torsion subsheaf, hence is pure.

Suppose otherwise. Then because $\mathcal{G}_0^i \cap \mathcal{B}_0^{(i+1)} = 0$, we have an infinite chain of inclusions

$$\mathcal{G} = \mathcal{G}_0^{(0)} \hookrightarrow \mathcal{G}_0^{(1)} \hookrightarrow \cdots$$

and

$$\mathcal{B} = \mathcal{B}_0^{(0)} \hookleftarrow \mathcal{B}_0^{(1)} \hookleftarrow \cdots .$$

We are assuming all of the $\mathcal{B}_0^{(i)}$'s have the same dimension and rank as those of $\mathcal{B}_0$.

By Lemma 5.7 (3), all the inclusions above are not isomorphisms. We notice that $P_H(\mathcal{G}_0^{(i+1)}) - P_H(\mathcal{G}_0^{(i)}) = P_H(\mathcal{B}_0^{(i)}) - P_H(\mathcal{B}_0^{(i+1)})$ and the dimension of $\mathcal{B}_0^{(i)}$ is smaller than the dimension of $\mathcal{G}_0^{(i)}$. Hence by our assumptions, the $\mathcal{G}_0^{(i)}$'s only differ in codimension ≥ 2, therefore they have the same reflexive hull $\mathcal{G}_0^{DD}$. Viewing $\mathcal{G}_0^{(i)}$ as an infinite increasing chain of subsheaves in $\mathcal{G}_0^{DD}$, we find a contradiction with the Noetherianness of $\mathcal{G}_0^{DD}$.

The converse part is easy. Suppose $G \subset F$ is a coherent subsheaf with support of lower dimension than $\dim(F)$. Then $\mathcal{G} := G \cap \mathcal{F}$ gives a contradiction to the assumption that $\mathcal{F}_0$ is pure.

Proof of (2). By the first part of our theorem and Lemma 5.5, we can choose a formal model $\mathcal{F}$ such that it is semistable and its special fiber $\mathcal{F}_0$ is pure on $\mathcal{X}_0$. Now we want to find a finitely presented subsheaf $\mathcal{F}' \subset \mathcal{F}$ with torsion quotient and special fiber $\mathcal{F}'_0$ semistable on $\mathcal{X}_0$. Starting from $\mathcal{F} = \mathcal{F}^{(0)}$, we will do induction on the semistable codimension of $\mathcal{F}_0$ which we denoted as $k(\mathcal{F}_0)$.

We denote the maximal destabilizing sheaf of codimension $k(\mathcal{F}_0) + 1$ in $\mathcal{F}_0$ by $\mathcal{B}_0$, and the quotient by $\mathcal{G}_0$. It is easy to see that $\mathrm{Hom}(\mathcal{B}_0, \mathcal{G}_0) = 0$. Then applying Lemma 5.7 to the short exact sequence $0 \to \mathcal{B}_0 \to \mathcal{F}_0 \to \mathcal{G}_0 \to 0$, we get a finitely

presented subsheaf $\mathcal{F}^{(1)} \subset \mathcal{F}$ and a short exact sequence $0 \to \mathcal{G}_0 \to \mathcal{F}_0^{(1)} \to \mathcal{B}_0 \to 0$. It is easy to see that $k(\mathcal{F}_0^{(1)}) \geq k(\mathcal{F}_0)$. After repeatedly applying the above procedure we will obtain a sequence of finitely presented sheaves

$$\mathcal{F} = \mathcal{F}^{(0)} \supset \mathcal{F}^{(1)} \supset \cdots .$$

Moreover, we will get short exact sequences

$$0 \to \mathcal{B}_0^{(i)} \to \mathcal{F}_0^{(i)} \to \mathcal{G}_0^{(i)} \to 0$$

and

$$0 \to \mathcal{G}_0^{(i)} \to \mathcal{F}_0^{(i+1)} \to \mathcal{B}_0^{(i)} \to 0.$$

We claim that after repeating the above procedure finitely many times $\mathcal{B}_0^{(i)}$, the maximal destabilizing sheaf of codimension $k(\mathcal{F}_0) + 1$, will have either rank or a_{k+1} smaller than that of $\mathcal{B}_0$. Recall that rank takes values in $\frac{\mathbb{N}}{n!}$ and a_{k+1} takes values in $\frac{\mathbb{Z}}{n!}$. Hence according to our claim, after N steps, $a_{k+1}(\mathcal{B}_0^{(N)}) = a_{k+1}(\mathcal{F}^{(N)})$. This implies $\mathcal{F}_0^{(N)}$ is semistable of codimension at least $k(\mathcal{F}_0) + 1$. By induction we would be done.

Now we prove our claim. Suppose all of the $\mathcal{B}_0^{(i)}$'s have non-decreasing rank and a_{k+1}. In that situation, by considering the injection $\frac{\mathcal{B}_0^{(i+1)}}{\mathcal{G}_0^i \cap \mathcal{B}_0^{(i+1)}} \hookrightarrow \mathcal{B}_0^{(i)}$ we see that $\mathcal{G}_0^i \cap \mathcal{B}_0^{(i+1)} = 0$. Therefore we may assume all of the $\mathcal{B}_0^{(i)}$'s have the same rank and a_{k+1} as those of $\mathcal{B}_0$. Hence we will get an infinite chain of inclusions

$$\mathcal{G} = \mathcal{G}_0^{(0)} \hookrightarrow \mathcal{G}_0^{(1)} \hookrightarrow \cdots$$

and

$$\mathcal{B} = \mathcal{B}_0^{(0)} \hookleftarrow \mathcal{B}_0^{(1)} \hookleftarrow \cdots .$$

By Lemma 5.7 (3), none of the inclusions above are isomorphisms. We note that $P_H(\mathcal{G}_0^{(i+1)}) - P_H(\mathcal{G}_0^{(i)}) = P_H(\mathcal{B}_0^{(i)}) - P_H(\mathcal{B}_0^{(i+1)})$ and by our assumption the $\mathcal{B}_0^{(i)}$'s only differ in codimension ≥ 2. Hence we see that the $\mathcal{G}_0^{(i)}$'s also only differ in codimension ≥ 2, therefore they have the same reflexive hull $\mathcal{G}_0^{DD}$. Viewing $\mathcal{G}_0^{(i)}$ as an infinite increasing chain of subsheaves in $\mathcal{G}_0^{DD}$, we find a contradiction with the Noetherianness of $\mathcal{G}_0^{DD}$.

The converse direction is the same as the proof of the first part. $\qquad\square$

Theorem 5.9. *In the situation above, if $\mathcal{F}^1$ and $\mathcal{F}^2$ are two formal models with semistable reductions, then their reductions $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ are S-equivalent.*

Proof. Suppose $\mathcal{F}^1$ and $\mathcal{F}^2$ are two formal models with semistable reductions. Then by Remark 4.3 we see that there is a decreasing filtration on $\mathcal{F}_0^1$ and an increasing filtration on $\mathcal{F}_0^2$, along with exact sequences

$$0 \to \mathrm{Fil}^{i+1}(\mathcal{F}_0^1) \to \mathrm{Fil}^i(\mathcal{F}_0^1) \to \frac{\mathcal{F}_0^2}{\mathrm{Fil}^i(\mathcal{F}_0^2)} \to \frac{\mathcal{F}_0^2}{\mathrm{Fil}^{i+1}(\mathcal{F}_0^2)} \to 0.$$

Because both $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ are semistable with the same Hilbert polynomial, by induction on the number of filtrations we see that $\mathrm{Fil}^i(\mathcal{F}_0^1)$, $\mathcal{F}_0^2 / \mathrm{Fil}^i \mathcal{F}_0^2$, $\mathrm{Gr}^i(\mathcal{F}_0^1)$ and $\mathrm{Gr}^i(\mathcal{F}_0^2)$ are all semistable with the same Hilbert polynomial. Now by Remark 4.3 we have isomorphisms between $\mathrm{Gr}^i(\mathcal{F}_0^1)$ and $\mathrm{Gr}^i(\mathcal{F}_0^2)$. Putting the above together, we see that $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ are S-equivalent. $\qquad\square$

Similarly one can prove the following theorem, which generalizes Langton's theorem to the case of arbitrary valuations.

Theorem 5.10. *Let $X \to \mathrm{Spec}(R)$ be a flat projective finitely presented scheme over a valuation ring R. Let F_K be a semistable sheaf on X_K. Then there exists a coherent sheaf F on X with generic fiber F_K and special fiber F_0 semistable on X_0. Moreover any two such coherent sheaves have S-equivalent special fibers.*

Proof. The proof is almost the same, the only subtlety being that a finitely presented algebra over a valuation ring is always coherent (c.f. [Gla89, Theorem 7.3.3]). □

Remark 5.11. One can use the above method to prove other semistable reduction type theorems. For example, semistable reduction for multi-filtered vector spaces or quiver representations. The former case has been worked out by the author in [Li17]. There is a slight subtlety as the category of multi-filtered vector spaces is not abelian.

Remark 5.12. Every line bundle is automatically stable, so by Theorem 5.8 one can always find a formal model of a given line bundle with a semistable reduction. This will be used in the last section to prove the main theorem.

6. Proof of the Main Theorem

6.1. **The Auxiliary Space W.** Assume $\mathcal{X}_0$ is a projective variety over k and fix an ample invertible sheaf H. Let P be the Hilbert polynomial of $\mathcal{O}_{\mathcal{X}_0}$. Then for every $\gamma \in \Gamma$ we denote the moduli stack of finitely presented sheaves with semistable geometric fiber of Hilbert polynomial P on $\mathcal{X}_\gamma / \mathcal{O}_\gamma$ by $\mathcal{M}_\gamma$. The existence of such a stack will be justified in the following, as we do not assume $\mathcal{X}_\gamma / \mathrm{Spec}(\mathcal{O}_\gamma)$ to be projective. Nevertheless we know that $\mathcal{M}_0$, the algebraic stack of semistable coherent sheaves of Hilbert polynomial P, is an open substack of $|Coh_{\mathcal{X}_0 / \mathrm{Spec}(k)}| = |Coh_{\mathcal{X}_\gamma / \mathrm{Spec}(\mathcal{O}_\gamma)}|$. Therefore we can define $\mathcal{M}_\gamma$ to be the corresponding open substack in $Coh_{\mathcal{X}_\gamma / \mathrm{Spec}(\mathcal{O}_\gamma)}$. For general theory concerning the stack of coherent sheaves, we refer to [Sta17, Tag 09DS].

By the description of the functors we know that $\mathcal{M}_\gamma \times_{\mathcal{O}_\gamma} \mathcal{O}_{\gamma'} = \mathcal{M}_{\gamma'}$ for any $\gamma' \le \gamma$, so the $\mathcal{M}_\gamma$'s form a projective system of algebraic stacks. Langer has shown that $\mathcal{M}_0$ is a quasi-compact algebraic stack (c.f. [Lan04, Theorem 4.4]). In particular, for a chosen pseudo-uniformizer $\pi \in \mathfrak{m}$ we can find a smooth surjection $\mathcal{W}_\pi \to \mathcal{M}_\pi$ where $\mathcal{W}_\pi = \mathrm{Spec}(A_1)$ is an affine scheme of finite presentation over $\mathcal{O}_\pi$. The following is a result of Emerton which is the key of this subsection.

Proposition 6.1 (Emerton, [Sta17, Tag 0CKI])**.** *Let $\mathcal{X} \subset \mathcal{X}'$ be a first order thickening of algebraic stacks. Let W be an affine scheme and let $W \to \mathcal{X}$ be a smooth morphism. Then there exists a cartesian diagram*

$$
\begin{array}{ccc}
W & \longrightarrow & W' \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X}'
\end{array}
$$

with $W' \to \mathcal{X}'$ smooth.

Applying this result yields the following commutative diagrams

$$
\begin{array}{ccc}
\mathcal{W}_i & \longrightarrow & \mathcal{W}_{i+1} \\
\downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} \\
\mathcal{M}_{\pi^i} & \longrightarrow & \mathcal{M}_{\pi^{i+1}}
\end{array}
$$

where $\mathcal{W}_i = \mathrm{Spec}(A_i) \hookrightarrow \mathcal{W}_{i+1} = \mathrm{Spec}(A_{i+1})$ is the thickening defined by π^i. Now $\mathcal{A}' = \varprojlim A_i$ is a topologically finitely generated $\mathcal{O}$-algebra, and $\mathcal{A} = \mathcal{A}'/(\pi\text{-torsions})$ is a topologically finitely presented $\mathcal{O}$-algebra by [BL93a, Proposition 1.1 (c)]; denote its formal spectrum $\mathrm{Spf}(\mathcal{A}) = \mathcal{W}$ and let $W = \mathcal{W}^{\mathrm{rig}}$ be its associated rigid analytic space which is an affinoid space. This is the auxiliary space we want. By the description of functors, we have a system of $\mathcal{W}_i$-flat coherent sheaves $\mathcal{F}_i$ on $\mathcal{X}_{\pi^i} \times_{\mathcal{O}_{\pi^i}} \mathcal{W}_i$. Therefore we get a $\mathcal{W}$-flat finitely presented sheaf $\mathcal{F} = (\varprojlim \mathcal{F}_i)/(\pi - torsions)$ (c.f. [BL93a, Lemma 1.2 (c)]) on $\mathcal{X} \times_{\mathcal{O}} \mathcal{W}$, and taking generic fiber gives us a W-flat coherent sheaf $F^{\mathrm{univ}} = \mathcal{F}^{\mathrm{rig}}$ on $X \times W$.

We can also define $R_i = W_i \times_{\mathcal{M}_{\pi^i}} W_i$, $\mathcal{R} = (\varprojlim R_i)/(\pi\text{-torsions})$ and $R = \mathcal{R}^{\mathrm{rig}}$. Note that we will get an equivalence relation $R \rightrightarrows W$.

Question 6.2. Can one make sense of "W/R" and prove it is the rigid stack of semistable coherent sheaves on X of Hilbert polynomial P?

6.2. **Determinant Construction.** In this subsection, we will explain the determinant construction which associates a flat family of coherent sheaves on a smooth proper rigid variety to a flat family of line bundles. This is well known to the experts and is written down in [KM76]. For reader's convenience we will briefly introduce the construction in the following.

The following lemma is a disguise of [Sta17, Tag 068x], and is the starting point of our determinant construction.

Lemma 6.3. *Let $X \to S$ be a smooth map of rigid spaces of relative dimension d, and let F be an S-flat coherent sheaf on X. Then F is perfect as a complex of coherent sheaves on X, i.e., we can find an admissible covering of X such that on each admissible open F can be resolved by locally free coherent sheaves. In fact the length of each resolution is at most d.*

Proof. We may reduce to the situation where $X = \mathrm{Sp}(B) \to S = \mathrm{Sp}(A)$ is smooth in rigid sense and F is given by a finitely generated B-module M. Now we meet every condition in [Sta17, Tag 068x] except for (2), but we have a replacement: for every maximal ideal $\mathfrak{n} \subset A$ the ring $B \otimes_A \kappa(\mathfrak{n})$ has finite global dimension $\leq d$. Because it suffices to check the local tor dimension of M is $\leq n$ at maximal ideals $\mathfrak{m} \subset B$ and every maximal ideal in $\mathrm{Spec}(B)$ is mapped to a maximal ideal in $\mathrm{Spec}(A)$, our replacement condition will make the original argument in the stacks project work. $\square$

Let $X \to S$ be a smooth proper map of rigid spaces of relative dimension d where S is an affinoid and F an S-flat coherent sheaf on X. Then we can find an admissible covering $\mathcal{U} = \{U_i = \mathrm{Sp}(A_i)\}$ of X such that on each U_i we can find a projective resolution of F of length at most d:

$$
K_i^\bullet \to F|_{U_i} \to 0.
$$

Then we define $\det(F)|_{U_i} = \bigotimes \det(K_i^j)^{\otimes(-1)^j}$, where by $\det(K)$ of a locally free sheaf K we mean its top rank self wedge product. Now on the overlap $U_{ij} = U_i \cap U_j$, we get two resolutions of $F|_{U_{ij}}$. So we get a quasi-isomorphism

$$\Phi_{ij} : K_i^\bullet|_{U_{ij}} \to K_j^\bullet|_{U_{ij}}$$

which induces a canonical isomorphism

$$\phi_{ij} : (\det(F)|_{U_i})|_{U_{ij}} \to (\det(F)|_{U_j})|_{U_{ij}}.$$

Moreover ϕ_{ij} only depends on the homotopy class of Φ_{ij} (c.f [KM76, Theorem 1 and Proposition 2]). On triple intersection $U_i \cap U_j \cap U_k$ the composition of chosen maps between resolutions $\Phi_{ki} \circ \Phi_{jk} \circ \Phi_{ij}$ is homotopic to the identity, hence the cocycle condition is satisfied automatically. Therefore the ϕ_{ij}'s give rise to gluing datum of $\det(F)|_{U_i}$. We just need the following two easy properties of this determinant construction.

Proposition 6.4.

(1) *Let L be a line bundle on X. Then we have a canonical isomorphism*

$$L = \det(L).$$

(2) *Let $f : T \to S$ be an arbitrary morphism of rigid spaces. Then we have a canonical isomorphism*

$$\det(f^*F) = f^*(\det(F)).$$

6.3. **Proof of the Main Theorem.** Before proving our Main Theorem, let us fix the notations. Let K' be a finite extension of K. Denote by $\{\}'$ the base change of corresponding objects from K (resp. $\mathcal{O}$) to K' (resp. $\mathcal{O}' = \mathcal{O}_{K'}$).

Proof of the Main Theorem 1.3. It suffices to prove that $\underline{\mathrm{Pic}}^P_{X/K}$ is a bounded functor by Theorem 3.5. Consider F^{univ} on $X \times W$ which is a W-flat coherent sheaf. The determinant construction gives us a W-flat line bundle on $X \times W$, hence a map $\det : W \to \underline{\mathrm{Pic}}^P_{X/K}$. We claim this map is a surjection on classical Tate points. This means that for every $P \in \underline{\mathrm{Pic}}^P_{X/K}(K')$ we can find a preimage of P in $W(\bar{K}')$, where K' is a finite extension of K with $\mathcal{O}' \cap K = \mathcal{O}$.

Indeed, P corresponds to a line bundle L on $X_{K'}$ with Hilbert polynomial the same as that of $\mathcal{O}_X$. Then by Theorem 5.8 we see that L has a formal model $\mathcal{F}'$ on $\mathcal{X}_{\mathcal{O}'}$ where $\mathcal{F}'_0$ is semistable. Note that $\mathcal{F}'_0$ automatically has Hilbert polynomial the same as that of $\mathcal{O}_{\mathcal{X}_0}$ by Theorem 4.2. $\mathcal{F}'$ gives rise to the maps s_i from bottom row to middle row in the diagram 6.1 below, where the subscript i here means the π^{i+1}-level of corresponding objects with π chosen as in Section 6.1. As the map $\mathcal{W}'_0 \to \mathcal{M}'_0$ is surjective and of finite presentation, after a further (unramified) finite extension of K' (for simplicity we will still call it K' below) we can lift the map s_0 to $\sigma_0 : \mathrm{Spec}(\mathcal{O}'/(\pi)) \to \mathcal{W}'_0$. By smoothness of f_i we can thus lift the maps s_i to

$\sigma_i : \mathrm{Spec}(\mathcal{O}'/(\pi'^{i+1})) \to \mathcal{W}'_i.$

(6.1)

Therefore we get a map $\sigma : \mathcal{O}' \to \mathcal{W}$, taking generic fiber gives us a K'-point $Q \in W(K')$. By Proposition 6.4 we see that $\det(Q) = P$. This completes our proof. $\qquad\square$

Remark 6.5. Our method actually proves $W \to \underline{\mathrm{Pic}}^P_{X/K}$ is a surjection on analytic (i.e. rank 1) points.

Proof of Theorem 1.1. Since $\mathrm{Pic}^0_{X/K}$ is a connected component of $\mathrm{Pic}^P_{X/K}$, it suffices to show the latter space is proper. This follows from our Main Theorem 1.3 and Proposition 3.2. $\qquad\square$

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