On rigid varieties with projective reduction

Shizhang Li Columbia University

Background

In a series of papers, Bosch, Lütkebohmert and Raynaud laid down the foundations relating formal and rigid geometry. The type of questions they treat in these papers are mostly concerned with going from the rigid side to the formal side. In this work we will consider the opposite type of question, namely we will investigate to what extent properties on the formal side inform us about rigid geometry.

Works of Others

1.A theorem of Huber–Lütkebohmert–Temkin asserts that a rigid space is proper iff (any of) its formal models are proper.

2. Let X be a connected smooth proper rigid space over a discretely valued non-archimedean field K. Hartl and Lütkebohmert proved the representability of the Picard functor on the category of smooth rigid spaces over K under an additional assumption that X has a strict semistable formal model. They also prove that in this situation the Picard group space always admits a Raynaud's uniformization (essentially they look like semi-abelian varieties).

3. In Warner's thesis it is proved that, assuming K has characteristic 0, the Picard functor defined on a suitable category of adic spaces over K is represented by a separated rigid space over $\mathrm{Spa}(K, \mathcal{O})$.

4. It is a consequence of Kedlaya–Liu that $\underline{\mathrm{Pic}}_{X/K}$ is always a partially proper functor.

Motivating Question

Let K be a non-archimedean field with residue field k. Let X be a connected smooth proper rigid space over K.

Question. Does X always acquire a formal model whose special fibre is projective?

Main Theorem

Theorem 1. *Suppose that X has a formal model $\mathcal{X}$ whose special fiber $\mathcal{X}_0$ is projective over $\mathrm{Spec}(k)$, and assume furthermore that the Picard functor is represented by a quasi-separated rigid space. Then Pic^0_X is proper.*

The Theorem above is deduced from the following:

Theorem 2 (Main Theorem)**.** *Suppose that X has a formal model $\mathcal{X}$ whose special fiber $\mathcal{X}_0$ is projective over $\mathrm{Spec}(k)$. Then $\underline{\mathrm{Pic}}^0_{X/K}$ is a proper functor.*

Proof Idea

1.

$$K_0(X) \longrightarrow K_0(\mathcal{X}_0)$$

$$[F] \longmapsto [\mathcal{F}_0]$$

A choice of an ample line bundle on $\mathcal{X}_0$ makes it possible to attach a Hilbert polynomial to any coherent sheaf F on X. Therefore we can define notions such as semistable sheaves on X.

2. We prove a generalization of Langton's theorem:

$$\text{semistable } F \rightsquigarrow \text{semistable } \mathcal{F}.$$

This implies that

$$[L] \in \mathrm{Pic}^0_X \rightsquigarrow \text{semistable } \mathcal{L} \text{ with same Hilbert polynomial}$$

3. Langer's theorem tells us that the reduction of these line bundles form a quasi-compact family. This is the starting point of the desired "finiteness" result.

Corollaries

Corollary 1. *Non-archimedean Hopf surfaces over a non-archimedean field have no projective reduction.*

Corollary 2 (joint with David Hansen)**.** *Let X be a proper smooth rigid variety over p-adic field. Assume X has a formal model whose special fiber is projective. Then we have*

$$h^{1,0}(X) = h^{0,1}(X).$$

A Further Question

Let X be a smooth proper rigid space over a characteristic 0 non-archimedean field admitting a formal model with projective reduction. Is it true that $h^{i,j}(X) = h^{j,i}(X)$ for all i, j?