# Weakly étale v.s. pro-étale.

I. Ind/Weakly-étale  II. Statements and a consequence  III. pf.

I. ~~Fix~~ $A \xrightarrow{f} B$ ring map is called:

Defn. • ind-étale if $B$ is ~~of~~ filtered colimit of étale $A$-alg.

"they are stable under composition and base change" • weakly-étale if $A \to B$ is flat & $B \otimes_A B \twoheadrightarrow B$ is also flat.

Defn. • $I \subseteq R$ ideal is called pure if $R/I$ is flat-$R$-alg.

~~Prop~~ Lemma. If $I \subseteq R$ is a pure ideal, then $\forall J \subseteq R$ ideal, $I \cdot J = I \cap J$.

pf: Consider $\left[(0 \to J \to R \to R/J \to 0) \otimes_R R/I\right] = 0 \to \frac{J}{IJ} \to \frac{R}{I} \to \frac{R}{I+J} \to 0$

In fact, $L_{B/A} \simeq 0$, page 9.

Cor. • ~~Apply to~~ If $A \to B$ is weakly étale, then $\Omega^1_{B/A} = 0$, hence it's formally unramified.

Ex: $k \to \prod_{\mathbb{N}} k$ is weakly étale iff $k$ is a finite field.

• weakly étale + finite presentation = étale.

Lemma ~~If f is ind-étale, it's then it's we~~ Ind-étale $\Rightarrow$ weakly étale.

pf. ~~colimit of flat maps is flat~~

consider:

$$\begin{tikzcd} B \otimes_A B \arrow[r] \arrow[d, equal] & B \\ \operatorname{colim} B_i \otimes_A B_i \arrow[r, "\text{flat}"] & B_i \arrow[u, "\text{flat}"'] \end{tikzcd}$$

Lemma.

$$\begin{tikzcd} B \arrow[rr, "g"] & & C \\ & A \arrow[ul, "f"] \arrow[ur, "g\circ f"'] & \end{tikzcd}$$

, if $f$ & $g \circ f$ are $\begin{cases} \text{ind-} \\ \text{weakly} \end{cases}$ étale, then $g$ is $\begin{cases} \text{ind-} \\ \text{weakly} \end{cases}$ étale.

pf. • ind-étale case can be reduced to étale case

• weakly-étale case:

$$\begin{tikzcd} B \arrow[r] & B \otimes_A C \arrow[r] & C & \\ & & B \otimes_A B \arrow[ul] \arrow[r] & B \arrow[ul] \\ A \arrow[uu] \arrow[r] & C \arrow[uu] & & \end{tikzcd}$$

$$\begin{tikzcd} & C & \\ C \otimes_A C \arrow[ur, "\text{flat}"] \arrow[rr, two heads, "\text{surj.}"] & & C \otimes_B C \arrow[ul, "\text{flat}"'] \\ B \otimes_A B \arrow[u] \arrow[rr, two heads] & & B \arrow[u] \end{tikzcd}$$

until (field).

*Lemma:*

$$\begin{tikzcd} B \arrow[rr, "f.f."] & & C \\ & A \arrow[ul, "w\text{-}ét"] \arrow[ur, "w\text{-}ét"'] & \end{tikzcd}$$

pf. commutative diagram:

$$\begin{tikzcd} C\otimes_A C \arrow[r, "flat"] & C \\ B\otimes_A B \arrow[u, "flat"] \arrow[ur, "\Rightarrow flat"] \arrow[r, "flat"'] & B \arrow[u, "f.f."'] \end{tikzcd}
\qquad
\begin{tikzcd} B \arrow[rr, "f.f."] & \Leftarrow & C \\ & A \arrow[ul, "flat"] \arrow[ur, "flat"'] & \end{tikzcd}$$

Defn. • $0\le d\in\mathbb{Z}$, we say $A$ has weak dim'n $\le d$ if every $A$-module has tor dim'n $\le d$. (equivalently, admits flat resolution of length $d$)

• $A$ has weak dim'n $\le 0$ is ~~sai~~ also called absolutely flat

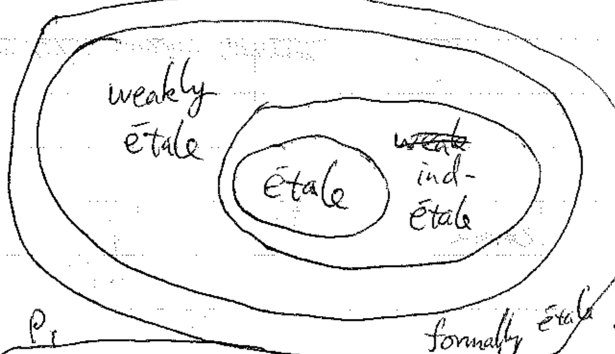
"all the modules are flat". (equivalent to $A$ being ① reduced ② 0-dim'l)

Fact: $\left.\begin{array}{l} A\longrightarrow B \text{ weakly étale} \\ A \text{ has weak dim'n} \le d \end{array}\right\} \Rightarrow B$ has weak dim'n $\le d$.

"pf: $(M\otimes_A N)\otimes_{B\otimes_A B} B = M\otimes_B N$.

II. Main Thm:

Thm A. $f: A\to B$ weakly étale. Then $\exists$ $B\longrightarrow C$ faithfully flat and ind-étale s.t. $A\longrightarrow C$ is ind-étale.




Cor. Weakly étale & pro-étale gives rise to the same ~~two topoi~~ Topoi on the category of Affine Schemes:

pf:

$$\begin{tikzcd} U \arrow[r, "w\text{-}ét"] & V \\ W \arrow[u, "pro\text{-}ét"] \arrow[ur, "pro\text{-}ét"'] & \end{tikzcd}
,\qquad
\begin{tikzcd} W\times_V W \arrow[r] \arrow[d] \arrow[rr, bend left, "p_1"] \arrow[dd, bend right, "p_2"'] & W\times_V U \arrow[r] \arrow[d] & W \arrow[d] \\ U\times_V W \arrow[r] \arrow[d] & U\times_V U \arrow[r, "p_1"] \arrow[d, "p_2"] & U \\ W \arrow[r] & U & \end{tikzcd}$$

It's easy to see that Thm A implies:

Thm B: $A$ is a strictly henselian local ring, $B$ is a weakly étale local $A$-alg. Then $f: A \to B$ is an isom.

Thm C: $K$ is a field, Any weakly étale $K$-alg. $A$ is ~~a direct limit of~~ ind-étale$/K$.

Example: $k \longrightarrow \prod_{\mathbb{N}} k$ is weakly étale iff $k$ is a finite field.

plan: Thm B $\Rightarrow$ Thm A, $*$ ~~Th~~ pf of Thm C,
$*$ Thm C $\Rightarrow$ Thm B.

~~proof~~ III. "Let's do a warm-up":

Pf of Thm C (maybe later?): It suffices to show any ~~element~~ f.g. $K$-sub-alg. $A'$ is étale$/K$.

~~Go to top of page 2, implies~~

$$A \text{ has weak dim'n } 0 \Longrightarrow (f) = (f)^2 \ \ \forall f \in A \ (\text{since } (f) \text{ is a pure ideal}).$$

$$\Downarrow$$

every local ring of $A$ is a field. $\Leftarrow$ $A$ is reduced & every prime is max'l. $\Leftarrow$ $(f) = (e)$ where $e$ is an idempotent

$K \hookrightarrow A' \hookrightarrow A$, $\forall$ minimal prime $\mathfrak{p}' \subseteq A'$, $\exists$ ~~minimal~~ prime $\mathfrak{p} \subseteq A$ above it.

~~hence~~ consider

$$\begin{tikzcd} K \arrow[r] \arrow[rr, bend right, "\text{still weakly étale}"'] & \kappa(\mathfrak{p}') \arrow[r] & \kappa(\mathfrak{p}) \cong A_{\mathfrak{p}} \end{tikzcd}$$

Go to top page 2, implies $K \longrightarrow \kappa(\mathfrak{p}')$ is weakly étale but it's also f.p. hence étale. (sep. alg. extn)

~~So every generic pt of Spec$(A')$ is closed~~, now $A'$ is finite type$/K$, reduced w/ ~~all~~ every generic pt are sep. alg. extn of $K$ ▨.

Thm C $\Rightarrow$ Thm B $\overset{?}{\Rightarrow}$ Thm A: Defn: ~~A topological space X is w-local if~~

- A ring $A$ is w-local if (1) every conn'd component of $\mathrm{Spec}(A)$ contains a unique closed point
  (2) the set of closed points $\mathrm{Spec}(A)^c$ is itself a closed subset of $\mathrm{Spec}(A)$.
- A ring homomorphism of w-local rings $A \to B$ is w-local if

A, B w-local, $A \to B$ is w-local ~~$f^{-1}$(max'l ideals in B) are max'l~~
$f^{-1}$(max'l ideal) is always a max'l ideal in $A$.

- A ~~w-local~~ ring $A$ is w-strictly local if
  (1) it's w-local and ~~all of~~ any $A \to B$ fppf étale admits a section.
  (2) ~~all the local rings @ closed pts are strictly henselian~~

{Lemma. A is a w-local ring, then
(1) any Zariski cover $\coprod U_i \longrightarrow \mathrm{Spec}(A)$ admits a section.
(2)
$$\begin{tikzcd}(\mathrm{Spec}(A))^c \arrow[r] \arrow[rr, bend right, "\text{homeomorphism}"'] & \mathrm{Spec}(A) \arrow[r] & \pi_0(\mathrm{Spec}(A))\end{tikzcd}$$

* pf*: (1) since any open cover of w-local is equiv. to an open cover of $\pi_0(\mathrm{Spec}(A))$ which is profinite.
(To see $\pi_0(\mathrm{Spec}(A))$ is always profinite, use Hochster's Thm: $\mathrm{Spec}(A)$ is always a cofiltered limit of finite $T_0$ space).
(2) by assumption: $\mathrm{Spec}(A)^c \longrightarrow \pi_0(\mathrm{Spec}(A))$ is a bijective continuous map between compact Hausdorff spaces.
(an affine scheme is 0-dim'l $\Leftrightarrow$ Hausdorff). hence a homeo...

Fact. ~~Lemma.~~ The inclusion of {w-local rings} $\longrightarrow$ {rings} admits a left adjoint
$$A^Z \longleftarrow A$$
where $A \to A^Z$, counit, is an ind-(Zariski localization).

Lemma: w-local ring $A$ is w-strictly local iff all local rings of $A$ at closed points are strictly henselian.

*pf* if $A$ is w-strictly local. $A_x \xrightarrow[\text{étale}]{\text{f.f.}} B'$ w/ $f$ invertible at $x$ ($= m$).

$$\begin{array}{ccc} A_x & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A[\frac{1}{f}] & \longrightarrow & B \end{array}$$

equivalently $(f) + m = A$.

hence $\exists\, h \in m$ w/ $h + fg = 1$. Consider $A \xrightarrow[\text{étale}]{\text{f.f.}} B \times A[\frac{1}{h}]$.

hence get $A_x \longrightarrow B_x' \times A'$, think about the image of $(0, h)$ has to be $0$, hence factors thru

Conversely, $A \xrightarrow[\text{ét}]{\text{f.f.}} B$, at every closed pt $x \in \mathrm{Spec}(A)$, $B_x$.

~~get~~ have section locally around each closed $x \in \mathrm{Spec}(A)$.

$\therefore \exists$ a Zariski cover of $\mathrm{Spec}(A)$, over which $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has a section. By w-locality, we find a section $B \to A$.

$\Big[\; B \longrightarrow A[\frac{1}{f_i}] . \quad B \longrightarrow \prod_i A[\frac{1}{f_i}] \leftarrow A \;\Big] \Big\}$.

Key Lemma: (write on the back or side board).

Let $f: A \to B$. Then $\exists$

$$\begin{array}{ccc} B & \dashrightarrow & B' \\ \uparrow & & \uparrow \\ A & \dashrightarrow & A' \end{array}$$

w/

① $A \to A'$ & $B \to B'$ are ~~f.f.~~ faithfully flat and ind-étale,

② $A'$ & $B'$ are strictly w-local

③ $A' \longrightarrow B'$ is w-local, inducing homeomorphism

~~local rings of~~ $\mathrm{Spec}(B')^c \cong \mathrm{Spec}(A')^c$.

~~④ $A'$ and $B'$ have strictly hen~~ ~~at closed pts are strictly henselian.~~

Thm C ⇒

pf of Thm B ⇒ Thm A:

$$\begin{tikzcd} B \arrow[r, "\text{ind-ét}", "\text{f.f}"'] & B' \\ A \arrow[u, "\text{w-ét}"] \arrow[r, "\text{ind-ét}"', "\text{f.f.}"] & A' \arrow[u] \end{tikzcd}$$

Conditions imply: $A' \longrightarrow B'$ is weakly étale, $\forall\ m \subseteq A'$, $B'/mB'$ is weakly étale over $A'/m$, hence ind-étale, but having a

(Thm C) ⇒ unique closed pt, hence $mB'$ is the unique max'l ideal above $m$.

Consider $A'_m \longrightarrow B'_{mB}$, it's an isom by $\boxed{\text{Thm B}}$

As all the ~~clos~~ max'l ideal in $B'$ are of the form $mB'$, we see that ~~$B'$~~ $A' \longrightarrow B'$ is also bijection on pts.

Hence $B'_{mB'} \xleftarrow{\simeq} B'_m$, therefore $A' \simeq B'$. ∎

Now back to page 4.

$$\begin{tikzcd} B \arrow[r] & B^Z \\ A \arrow[u] \arrow[r] & A^Z \arrow[u] \end{tikzcd}$$

these are w-local, w/ Jacobson ideal cutting out the closed set of closed pts.

Constructing $A'$ & $B'$

$$\begin{tikzcd} B^Z \arrow[r] & B^Z/J_B = B_0 \\ A^Z \arrow[u] \arrow[r] & A^Z/J_A = A_0 \arrow[u] \end{tikzcd}$$

Lemma. For $A_0 \longrightarrow B_0$ map of absolutely flat rings, $\exists$ ind-étale $A_0 \to \overline{A_0}$, $B_0 \to \overline{B_0}$

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ \overline{A_0} \arrow[r] & \overline{B_0} \end{tikzcd}$$

s.t. $\overline{A_0}$ & $\overline{B_0}$ are w-strictly local.

pf. $\overline{A_0} := \operatorname{colim}_{J \subseteq I} \cancel{A_J} \bigotimes_{j \in J} A_j$, where $J$ finite subset of $I :=$ set of isom classes of f.f. étale $A_0$-algebras

Do the same construction for $\overline{A_0} \otimes_{A_0} B_0$

$$\begin{tikzcd} B \arrow[r] & B^z \arrow[r] & B_0 \arrow[r] & \bar{B}_0 \\ A \arrow[u] \arrow[r] & A^z \arrow[u] \arrow[r] & A_0 \arrow[u] \arrow[r] & \bar{A}_0 \arrow[u] \end{tikzcd}$$

Construction: (Henselization) $\mathrm{Hens}_{A^z}(\bar{A}_0) = \mathrm{colim}\, A'$ where colimit is indexed by

$$A^z \xrightarrow{\text{étale}} A' \longrightarrow \bar{A}_0 .$$

[This is a functor: $\mathrm{Ind}(A_{0,\text{ét}}) \longrightarrow \mathrm{Ind}(A^z_{\text{ét}})$ s.t.

$$\mathrm{Hom}_{A_0}(\tilde{A} \otimes_{A^z} A_0, \bar{A}_0) = \mathrm{Hom}_{A^z}(\tilde{A}, \mathrm{Hens}_{A^z}(\bar{A}_0)) \,]$$

Lemma: $\mathrm{Hens}_{A^z}(\bar{A}_0) \otimes_{A^z} A_0 = \bar{A}_0$

pf: follows from the fact that any étale $A_0$ algebra ~~has a~~ lifts to an étale $A^z$-algebra.

Lemma: $\mathrm{Hens}_{A^z}(\bar{A}_0)$ is w-strictly local.

* pf:* $\mathrm{Hens}_{A^z}(\bar{A}_0) \xleftarrow{\text{ind-étale}} A^z$, ~~hence all Spec( max'l ideals of $\mathrm{Hens}_{A^z}(\bar{A}_0)$ contains $J_A$, and since~~ $\mathrm{Spec}(\mathrm{Hens}_{A^z}(\bar{A}_0))^c \simeq \mathrm{Spec}(\bar{A}_0)$

Check:

$x^2+(2f-1)x+j=0$

$2x+(2f-1)=0$

$\Downarrow$

$x^2-2x^2+j=0$

$\Downarrow$

$-x^2+j=0$.

but $2x+(2f-1)=0$

$\Downarrow$

$x$ is a unit

a topological argument shows it's w-local. No f.f. étale cover follows from functoriality.

[ $R \supseteq J$ = Jacobson, $R \twoheadrightarrow R/J$ is henselian. Then idempotents of $R$ and idempotents of $R/J$ are bijective.

if $f^2=f$, $(f+j)^2 = f+j \Rightarrow 2fj + j^2 - j = 0$.

$(2f-1)^2 = 1 \Rightarrow j(\underbrace{2f-1+j}_{\substack{2f-1 \,+\, j \text{ is a unit} \\ \text{unit} \quad \text{Jacobson}}}) = 0 \Big\} \Rightarrow j=0$. Injectivity.

$\bar{f}^2 = \bar{f} \Rightarrow f^2 = f+j$. Want $(f+j')^2 = f+j'$ ⇝ want $j' \in J$ s.t.

$(f+j')^2 = f+j+2fj'+j'^2$

$j'^2+(2f-1)j'+j=0$. But now $x^2+(2f-1)x+j$ defines an étale R-alg. which has a solution $\bar{0}$ mod. J.

$$\begin{tikzcd}
 & & \mathrm{Hens}_{B^Z}(\bar{B}_0) \arrow[r] & \bar{B}_0 \\
B \arrow[r] & B^Z \arrow[ur] \arrow[r, two heads] & B_0 \arrow[ur] & \\
 & & \mathrm{Hens}_{A^Z}(\bar{A}_0) \arrow[uu] \arrow[r] & \bar{A}_0 \arrow[uu] \\
A \arrow[uu] \arrow[r] & A^Z \arrow[uu] \arrow[ur] \arrow[r, two heads] & A_0 \arrow[ur] \arrow[uu] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r] & B' = \mathrm{Hens}_{B^Z}(\bar{B}_0) \\
A \arrow[u] \arrow[r] & A'' = \mathrm{Hens}_{A^Z}(\bar{A}_0) \arrow[u]
\end{tikzcd}$$

almost there... Check this satisfies all the conditions except for ③ $\mathrm{Spec}(B')^c \longrightarrow \mathrm{Spec}(A'')^c$ ~~is not~~

we have no controll so far!

Lemma/Construction: Any w-local map $f: A'' \longrightarrow B'$ of w-local rings admits a canonical factorization $A'' \xrightarrow{g} A' \xrightarrow{h} B'$ w/

① $A'$ w-local

② $g$ is a w-local ind-(Zariski localization).

③ $h$ is a w-local map inducing $\pi_0(\mathrm{Spec}(B')) \simeq \pi_0(\mathrm{Spec}(A'))$

pf.

$$\begin{tikzcd}
\mathrm{Spec}(B')^c \arrow[r] \arrow[dd] & \mathrm{Spec}(B') \arrow[r] \arrow[d] & \pi_0(\mathrm{Spec}(B')) \arrow[dd, "\text{pro-(Zariski localization)}"] \\
 & \mathrm{Spec}(A') = \cancel{\mathrm{Spe}}\, Z \arrow[ur] \arrow[d] & \\
\mathrm{Spec}(A'')^c \arrow[r] & \mathrm{Spec}(A'') \arrow[r] & \pi_0(\mathrm{Spec}(A''))
\end{tikzcd}$$

any map between profinite sets is pro-(Zariski localization).

Overview:

$$\begin{tikzcd}
 & & \mathrm{Hens}_{B^Z}(\bar{B}_0) \arrow[r] & \bar{B}_0 \\
B \arrow[r] & B^Z \arrow[ur] \arrow[r, two heads] & B_0 \arrow[ur] & \\
 & A' \arrow[uur] & & \\
 & & \mathrm{Hens}_{A^Z}(\bar{A}_0) \arrow[uuu] \arrow[ul] \arrow[r] & \bar{A}_0 \arrow[uuu] \\
A \arrow[uuu] \arrow[r] & A^Z \arrow[uuu] \arrow[ur] \arrow[r, two heads] & A_0 \arrow[ur] \arrow[uuu] &
\end{tikzcd}$$

OK, too tired. I don't wanna say how to get

Thm B from Thm C...

Lemma. If $A \to B$ is weakly étale, then $\mathbb{L}_{B/A} \simeq 0$.

pf. $B \otimes_A B \twoheadrightarrow B$ w/ $\ker = \ker^2 \Rightarrow \mathbb{L}_{B/B\otimes_A B} \simeq 0$.

$A \to B\otimes_A B \twoheadrightarrow B$ gives: $\mathbb{L}_{B/A} \simeq \mathbb{L}_{B\otimes_A B/A} \otimes_{(B\otimes_A B)} B \cdots ①$.

$$\begin{tikzcd} B \arrow[r] & B\otimes_A B \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$
gives: $\mathbb{L}_{B/A} \otimes_B (B\otimes_A B) \simeq \mathbb{L}_{B\otimes_A B/B} \cdots ②$.

and

$$\begin{tikzcd} \mathbb{L}_{B\otimes_A B/A} \arrow[rr] & & \mathbb{L}_{B\otimes_A B/B} \arrow[dl] \\ & \mathbb{L}_{B/A} \otimes_B (B\otimes_A B) \arrow[ul, "{[1]}"] & \end{tikzcd} \cdots ③$$

~~Tensor~~ ③ $\otimes_{(B\otimes_A B)} B$, combine ① & ②:

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/A} \quad \text{triangle} \Rightarrow \mathbb{L}_{B/A} \simeq 0.$$

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Thm C $\Rightarrow$ Thm B: $A \to B$ local hom. of local rings, weakly étale and $A$ is s.h.

Claim: for all $\mathfrak{p} \subseteq A$, $\exists!\ \mathfrak{q} \subseteq B$ above it w/ $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$.

This implies $B\otimes_A B \to B$ is bijective on Spec and flat.

Hence kernel = pure & nilpotent ideal $= 0$ ($\forall x \in I$, $x = xy$ w/ $y \in I$ $\Rightarrow x = xy^N = 0$ since $y$ is a nilpotent.)

But then $B \cong B\otimes_A B$, and $A \xrightarrow{f.f.} B$, hence $A \cong B$.

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To prove the claim, by Thm C, $B \otimes_A \kappa(\mathfrak{p})$ is ind-étale over $\kappa(\mathfrak{p})$.

Suppose the claim is wrong, then $\exists\ L/\kappa(\mathfrak{p})$ alg. sep. field extn, s.t. $B \otimes_A L$ has a nontrivial idempotent.

Now suppose such $L$ exists, consider $A' =$ int'l closure of $A/\mathfrak{p}$ in $L$.

$A'$ being integral over $A$ (s.h.) and domain, must be s.h. also.

~~And~~ $A \longrightarrow A'$ is integral and induces purely inseparable extn ~~in~~ on residue fields.

Hence: $B \otimes_A A'$ is also a ~~o~~ local ring.

Lemma/Fact: $A'$ is a normal ~~ring~~ domain w/ fraction field $L$, and $A' \hookrightarrow B'$ is weakly étale. Then $B'$ is int'lly closed in $B' \otimes_{A'} L$.

We apply this Lemma to $A', L$ and $B' = B \otimes_A A'$.

$\Rightarrow$ $B \otimes_A A'$ is integrally closed in $B \otimes_A L$

$\Rightarrow$ $\nexists$ ~~∄~~ nontrivial idempotents in $B \otimes_A L$, as $B \otimes_A A'$ is a local ring.