The Geometric Satake Equivalence II

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§1. Geometric Satake (part 2)

Where we are now

Recall from last time:

- G is a reductive group over the algebraically closed field k.
- Sat_G is the category of L^+G-equivariant $\overline{\mathbf{Q}}_\ell$-perverse sheaves on the affine Grassmannian Gr_G.
- We have shown that Sat_G is a rigid tensor category with fiber functor H^* (sum of all cohomology groups).
- By the Tannakian formalism, Sat_G is equivalent to $\mathsf{Rep}(\Gamma)$ for some affine group scheme $\Gamma/\overline{\mathbf{Q}}_\ell$.
- We proved that Γ is finite type, connected, and reductive.
- Main goal: show that Γ is the Langlands dual group $G^\vee$.

Braden's localization

- Let X be a variety on which $\mathbf{G}_m$-acts.
- Let X_0 be the $\mathbf{G}_m$-fixed locus.
- Let $X_0^1, \ldots, X_0^n$ be the irreducible components of X_0. Let X_+^i be the points $x \in X$ such that ux limits into X_0^i as $u \to 0$. Let X_+ be the disjoint union of the X_+^i.
- Define X^- similar, but for $u \to \infty$.
- Let $i_\pm \colon X_\pm \to X$ be induced by the inclusions and $q_\pm \colon X_\pm \to X_0$ the limit maps.
- Braden's theorem: have isomorphism $(q_+)_! i_+^* \mathcal{F} \cong (q_-)_* i_-^! \mathcal{F}$ for equivariant perverse sheaves $\mathcal{F}$.

The $\mathbf{G}_m$ action on Gr

- Fix a maximal torus T and Borel B of G such that $T \subset B$, and let U be the radical of B.
- Let $2\rho^\vee$ be the sum of the positive co-roots. This defines a homomorphism $\mathbf{G}_m \to T$, and thus an action of $\mathbf{G}_m$ on Gr_G through the action of $\mathrm{L}G$.
- The $\mathbf{G}_m$-fixed locus is the discrete set consisting of the points t^λ with λ an arbitrary coweight; this is identified with Gr_T.
- The X_+^λ locus is the $\mathrm{L}U$ orbit of t^λ; call this S_λ. Thus X_+ is identified with Gr_B. The maps $i \colon \mathrm{Gr}_B \to \mathrm{Gr}_G$ and $q \colon \mathrm{Gr}_B \to \mathrm{Gr}_T$ are induced by $B \to G$ and $B \to T$.
- Similarly, X_- is identified with Gr_{B_-} where B_- is the opposite Borel.

An example

Suppose $G = \mathbf{GL}(n)$. Then, up to some renormalization, $2\rho^\vee$ takes $u \in \mathbf{G}_m$ to the diagonal matrix $[u^n, u^{n-1}, \ldots, 1]$.

If we conjugate $g \in \mathbf{GL}(n)$ by this matrix, the upper triangular part of g is scalaed by positive powers of u, the lower triangular part by negative powers, and the diagonal is fixed.

Thus if g is upper-triangular then the conjugate matrix converges to a diagonal matrix as $u \to 0$.

This proves that the S_λ loci are contained in the X_+ set.

Cohomology on S_λ

Proposition

Let $\mathcal{A} \in \mathrm{Sat}_G$. Then $\mathrm{H}_c^*(S_\lambda, \mathcal{A})$ is concentrated in degree $(2\rho, \lambda)$.

Proof

- Computing the dimension of $S_\lambda \cap \mathrm{Gr}_\mu$, and bounding the amplitude of the restriction of the complex $\mathcal{A}$ to this locus, one shows that $\mathrm{H}_c^i(S_\lambda \cap \mathrm{Gr}_\mu, \mathcal{A})$ vanishes for $i > (2\rho, \lambda)$.
- The loci $S_\lambda \cap \mathrm{Gr}_\mu$ stratify S_λ. A spectral sequence argument now shows that $\mathrm{H}^i(S_\lambda, \mathcal{A}) = 0$ for $i > (2\rho, \lambda)$.
- Note that $\mathrm{H}_c^*(S_\lambda, \mathcal{A})$ is the λ component of $q_! i^* \mathcal{A}$.
- A dual argument shows that the cohomology of $(q_-)_*(i^-)^! \mathcal{A}$ vanishes in degrees $i < (2\rho, \lambda)$.
- The result now follows from Braden's theorem.

Cohomology on S_λ (cont)

Proposition

For $\mathcal{A} \in \mathsf{Sat}_G$ *we have* $\mathrm{H}^*(\mathcal{A}) = \bigoplus_\lambda \mathrm{H}^*_c(S_\lambda, \mathcal{A})$.

Proof

- The S_λ stratify Gr, and so we get a spectral sequence with E_1 page $\mathrm{H}^*_c(S_\lambda, \mathcal{A})$ converging to $\mathrm{H}^*(\mathcal{A})$.
- It degenerates at the E_1 page for the degree reasons in the previous slide.
- We thus have a natural filtration of $\mathrm{H}^*(\mathcal{A})$ with graded pieces $\mathrm{H}^*_c(S_\lambda, \mathcal{A})$.
- There is a dual filtrations with graded pieces $\mathrm{H}^*(S^-_\lambda, (i^-)^!\mathcal{A})$.
- These filtrations are complementary by Braden's theorem and thus canonically split.

Cohomology on S_λ (cont)

Proposition

The space $\mathrm{H}_c^*(S_\lambda, \mathrm{IC}_\mu)$ *has a natural basis indexed by the irreducible components of* $S_\lambda \cap \mathrm{Gr}_{\leq\mu}$*. It is non-zero only for* $\lambda \leq \mu$*, and is one-dimensional for* $\lambda = \mu$*.*

Proof

This also follows from properties of $S_\lambda \cap \mathrm{Gr}_\mu$. (E.g., for the second statement, the intersection is empty unless $\lambda \leq \mu$, and for $\lambda = \mu$ it is an affine space of dimension $(2\rho, \lambda)$.)

The dual torus

- Let $T^\vee$ be the torus dual to T. Thus the weights $X^\bullet(T^\vee)$ of $T^\vee$ are the coweights $X_\bullet(T)$ of T.
- Giving a representation of $T^\vee$ on a vector space is the same as giving a grading by $X_\bullet(T)$.
- We thus see that $\mathrm{H}^*(\mathcal{A})$ has a natural representation of $T^\vee$.
- This defines a functor $\mathrm{Rep}(\Gamma) \to \mathrm{Rep}(T^\vee)$, which one proves is symmetric monoidal.
- The Tannakian formalism gives a group homomorphism $T^\vee \to \Gamma$.

The dual torus is maximal

Proposition

$T^\vee$ is a maximal torus of Γ.

Proof

- Let $L_\mu = \mathrm{H}^*(\mathrm{IC}_\mu)$; these are the irreducible reps of Γ.
- We have seen that μ is the heighest weight of $T^\vee$ on L_μ, and occurs with multiplicity one.
- Since the weights appearing in representations of Γ span $X^\bullet(T^\vee)$, it follows that $T^\vee \to \Gamma$ is injective.
- Since the $T^\vee$ characters of the L_μ are linearly independent (by the heighest weight property), it follows that $T^\vee$ is maximal.

Completion of the proof

Theorem

We have $\Gamma = G^\vee$.

Proof

- Notion of dominance for coweights gives Borel subgroup of Γ.
- General fact: a weight λ of $T^\vee$ is a sum of positive roots if and only if $\mu - \lambda$ appears as a weight in L_μ for some μ.
- By the result on $\mathrm{H}_c^*(S_\lambda, -)$, this is equivalent to the existence of μ with $t^{\mu-\lambda} \in \mathrm{Gr}_{\leq\mu}$, or $\mathrm{Gr}_{\leq\mu-\lambda} \subset \mathrm{Gr}_{\leq\mu}$.
- By our previous analysis of Schubert varieties, this is equivalent to λ being a sum of positive coroots.
- A simple exercise with root data shows that $\Gamma = G^\vee$.

§2. Geometric Satake (part 3)

Richarz's proof

- Richarz gave a different proof of geometric Satake. We indicate the main idea.
- The proof starts the same way: Sat_G is equivalent to $\mathsf{Rep}(\Gamma)$ for some finite type, connected, reductive $\Gamma/\overline{\mathbf{Q}}_\ell$, and the problem is to show that $\Gamma = G^\vee$.
- For this, Richarz uses a theorem of Kazhdan, Larsen, and Varshavsky: if G and H are reductive groups then $G \cong H$ if and only if $\mathrm{K}_+(G) \cong \mathrm{K}_+(H)$, where K_+ denotes the Grothendieck semiring.
- Thus, it suffices to show $\mathrm{K}_+(\mathsf{Sat}_G) = \mathrm{K}_+(G^\vee)$.

Richarz's proof (cont)

To construct this isomorphism, Richarz appeals to the PRV (Parthasarathy–Ranga Rao–Varadarajan) conjecture, proved by S. Kumar:

Theorem (PRV conjecture)

Let G be a reductive group. Let λ and $\mu_1, \ldots, \mu_n$ be dominant weights such that $\lambda = w_1\mu_1 + \cdots + w_n\mu_n$ for some elements $w_1, \ldots, w_n$ of the Weyl group. Then L_λ appears as a summand in $L_{\mu_1} \otimes \cdots \otimes L_{\mu_n}$.

Richarz proves that a version of this holds in Sat_G, and that this is enough to identify $\mathrm{K}_+(\mathsf{Sat}_G)$ with $\mathrm{K}_+(G^\vee)$.

§3. Comparing geometric Satake and classical Satake

The classical Satake isomorphism

- Suppose now that k is a finite field and G is a split reductive group over k.
- Recall that the Hecke algebra $\mathcal{H}_G$ consists of the $G(\mathcal{O})$ bi-invariant $\mathbf{Z}$-valued functions on $G(F)$, where $F = k(\!(t)\!)$ and $\mathcal{O} = k[\![t]\!]$.
- The classical Satake transform gives an isomorphism $\mathcal{S}\colon \mathcal{H}_G \otimes \overline{\mathbf{Q}}_\ell \to \mathrm{K}(G^\vee) \otimes \overline{\mathbf{Q}}_\ell$, where $\mathrm{K}(G^\vee)$ denotes the Grothendieck group of $\mathrm{Rep}(G^\vee)$.
- We now compare this to the geometric Satake equivalence.

The Satake category

- One can define the Satake category Sat_G just as before.
- However, it is now "too big" since there is an extra Galois action.
- There is a notion of a "normalized" IC sheaf that takes into account the Galois action.
- The normalized IC sheaves span a tensor subcategory $\mathrm{Sat}_G^{\mathrm{N}}$ of Sat_G that is equivalent to the Satake category of G over $\overline{k}$.
- In particular, $\mathrm{Sat}_G^{\mathrm{N}}$ is equivalent to $\mathrm{Rep}(G^\vee)$ as a tensor category.

The sheaf-function dictionary

- Suppose X is a variety over k and $\mathcal{A}$ is a complex of ℓ-adic sheaves on X.
- We define a function $f_{\mathcal{A}} \colon X(k) \to \overline{\mathbf{Q}}_\ell$ by

$$f_{\mathcal{A}}(x) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(\mathrm{Frob}_x \,|\mathrm{H}^i(\mathcal{A}_x)).$$

- We can in particular apply this to objects in Sat_G. We thus get a well-defined map

$$f \colon \mathrm{K}(\mathrm{Sat}_G) \otimes \overline{\mathbf{Q}}_\ell \to \mathcal{H}_G \otimes \overline{\mathbf{Q}}_\ell$$

The sheaf-function dictionary (cont)

Proposition

The map f induces an isomorphism $\mathrm{K}(\mathsf{Sat}_G^{\mathrm{N}}) \otimes \overline{\mathbf{Q}}_\ell \cong \mathcal{H}_G \otimes \overline{\mathbf{Q}}_\ell$.

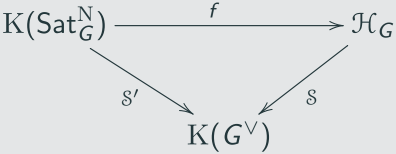
Proof

- It follows directly from the definitions that the convolution product of sheaves is compatible with the convolution product of functions under the sheaf-function dictionary.
- Let c_μ be the characteristic function of the double coset $G(\mathcal{O})t^\mu G(\mathcal{O})$. These elements (with μ a dominant coweight) form an integral basis of $\mathcal{H}_G$ by the Cartan decomposition.
- One sees that $f_{\mathrm{IC}_\mu} = c_\mu + \cdots$, where the omitted terms are smaller. This shows that f is an isomorphism.

The main result

Theorem

The following diagram commutes:



All groups should be tensored up to $\overline{\mathbf{Q}}_\ell$. *The map* $\mathcal{S}'$ *is the isomorphism on* K *groups induced by the geometric Satake equivalence.*

Proof

Compute explicitly using Kazhdan–Lusztig polynomials.

References

[R] T. Richarz. A new approach to the geometric Satake equivalence. arXiv:1207.5314

[Z] X. Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. arXiv:1603.05593