Pick 3 and solve.

(1) True or False: no need to copy the statements or explain, just write T or F.
(E.g. one could write (1): TTTTT.)

- Two sheaves F and G on a topological space X are isomorphic if and only if for each $x \in X$ there is a bijection $F_x \cong G_x$ between stalks.
- Let $f\colon X \to Y$ be a morphism of schemes, assume Y is reduced. Then f has dense image if and only if the map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.
- A scheme X is integral if and only if it is reduced and connected.
- Every nonempty scheme has a closed point.
- Let $X = Spec(\prod_{i=1}^\infty \mathbb{F}_3)$, it is locally Noetherian.

Answer is FTFFF.

(2) Let A be a ring, and let $\{f_i\}$ be a (not necessarily finite) set of elements in A generating unit ideal. Let M be an A-module, show that the following sequence

$$0 \to M \to \prod_i M[1/f_i] \to \prod_{i,j} M[1/f_i f_j]$$

is exact.

(3) Recall that a point x in a topological space X is said to be a generic point if its closure $\overline{\{x\}} = X$. If X is a scheme, show that every nonempty irreducible closed subset of X has a unique generic point.

(4) Show that a scheme X is affine if and only if there exists finitely many elements f_i in $A := \Gamma(X, \mathcal{O}_X)$ generating unital ideal and $X_{f_i} := \{x \in X \mid f_i \notin m_x \subset O_{X,x}\}$ is affine.

(5) Let X be a scheme locally of finite type over a field k. Show that the set of closed points of X is dense in X.

A morphism of schemes $X \to Y$ is called universal homeomorphism if for any morphism of schemes $Y' \to Y$ the pullback morphism $X \times_Y Y' \to Y'$ is a homeomorphism. The following questions concern this notion.

(6) If $X \to Y$ is a universal homeomorphism, assume $Y = Spec(k)$ where k is a field, and X is reduced, show that $X = Spec(k')$ where $k \subset k'$ is a purely inseparable field extension.

X is necessarily affine, so the map is affine and universally closed, hence must be integral. Therefore $X = Spec(k')$ for an algebra field extension. If it contains a nontrivial separable sub-extension, choose a finite degree such sub-extension $k \subset \ell$ and contemplate with base change along it yields a contradiction. Conversely, one can show that any purely-insparable extension is a universal homeomorphism on spectrum.

(7) If we have a factorization $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $g \circ f$ is a universal homeomorphism. Show that f is a universal homeomorphism if and only if g is a universal homeomorphism.

The only hint: The diagonal of a universal homeomorphism is an immersion and a homeomorphism, hence is a closed immersion corresponding to an ideal-sheaf affine locally given by nilpotents, therefore itself a universal homeomorphism.

(8) Let R be a DVR with uniformizer π. Consider the following scheme $X = Spec(R[x]) \setminus \{s\}$ where s is the maximal ideal (π, x). Show that X is not an affine scheme.

The ring of global function is seen to be $R[x]$, and the map *can* from below does not hit the maximal ideal (π, x).

Recall that given any scheme X, it admits a canonical morphism to an affine scheme $can\colon X \to Y := Spec(\Gamma(X, \mathcal{O}))$ coming from identity map on global section of structure sheaves. The following questions concern this map.

(9) If X is irreducible, show the map can has dense image.

(10) If X is quasi-compact, show the map can has dense image.

(11) Construct an example of X such that the map can does not have dense image.

For each natural number $n > 0$, let $A_n := k[x, y_1, y_2, \ldots, y_{n-1}, y_n^{\pm}, y_{n+1}^{\pm}, \ldots]/(x^i y_i)$. Notice that each $Spec(A_n) = D(y_n)$ in $Spec(A_{n+1})$. Let $X = \bigcup_{n>0} Spec(A_n)$, then the global function x is locally nilpotent but not so globally.