BRIEF NOTES ON AG II: CURVES

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1. Introducing the players

Let us fix a field k. In this section, we'll introduce examples of schemes (actually, varieties) that are known as "non-singular projective curves over k".

Construction 1.1 (Hyperelliptic curves)**.** Throughout this construction, we require that k has characteristic $\neq 2$. Let $P(x) \in k[x]$ be a polynomial of degree $d \geq 1$, such that $(P(x), P'(x))$ is the unit ideal. Below we shall define two affine schemes (which are smooth k-varieties), and glue them over an open, resulting a scheme called the "hyperelliptic curve associated with $P(x)$".

Let $U_1 := Spec\left(k[x,y]/(y^2 - P(x))\right)$. Denote the coordinate ring $A_1 := k[x,y]/(y^2 - P(x))$. This is our first affine scheme, simple, easy, yeah?

Now define a new polynomial $Q(x) \in k[x]$ by $Q(x) := P(1/x) \cdot x^{2\lceil d/2 \rceil}$. Let $U_2 := Spec\left(k[x,y]/(y^2 - Q(x))\right)$. Denote the coordinate ring $A_2 := k[x,y]/(y^2 - Q(x))$. This is our second affine scheme. Here's a fun exercise:

Exercise 1.2. (1) Check that indeed $Q(x) \in k[x]$;

(2) Show that $(Q(x), Q'(x))$ is again the unit ideal.

(3) Show that both U_1 and U_2 are smooth affine k-varieties.

(4) Show that there is a natural isomorphism of rings $A_1[1/x] \cong A_2[1/x]$ with $x \mapsto 1/x$ and $y \mapsto \frac{y}{x^{\lceil d/2 \rceil}}$.

Now we may glue U_1 with U_2 along opens $V_1 := Spec(A_1[1/x]) \subset U_1$ and $V_2 := Spec(A_2[1/x]) \subset U_2$, with the glueing isomorphism given by (4) above. The resulting scheme X is called the hyperelliptic curve associated with $P(x)$.

Here's a difficult exercise:

Exercise 1.3. (1) Show that X constructed above is a smooth k-variety. For instance, show that the map of k-algebra: $\mathcal{O}(U_1) \otimes_k \mathcal{O}(U_2) \to \mathcal{O}(V_1)$, where the map on the second factor is via $V_2 \subset U_2$ together with the isomorphism $\mathcal{O}(V_1) \cong \mathcal{O}(V_2)$, is surjective. (Make sure you understand why we need to show something like this.)

(2) What happens if we glue along a smaller affine open $V_1' \subset U_1$ with $V_2' \subset U_2$?

(3) What happens if characteristic of k is 2? Specifically, what can you say about the fraction field of $\left(k[x,y]/(y^2 - P(x))\right)/(\text{nilpotents})$ when $char(k) = 2$?

In the following exercise, we will compute the dimension of Hodge cohomology of hyperelliptic curves.

Exercise 1.4. Let k be a field of characteristic $\neq 2$. Let C be a hyperelliptic curve associated with polynomial $P(x) \in k[x]$.

(1) Compute the dimensions of $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$. (Hint: the first number is a constant regardless of $P(x)$, the second number depends solely on the degree of $P(x)$.)

(2) Let $A = k[x,y]/(y^2 - P(x))$, show that there is an element in $\Omega^1_{A/k}$ which deserves to be denoted $\frac{dx}{y} = \frac{2 \cdot dy}{P'(x)}$. (Hint: this element should have $y\times$ it being dx, and have $P'(x)\times$ it being $2dy$. In order to find this element, recall that $(P, P') = (1)$.)

(3) Show that in the setting of (2), we have $\Omega^1_{A/k} = A \cdot \frac{dx}{y}$. In other words, the rank 1 locally free A-module $\Omega^1_{A/k}$ is in fact free with generator $\frac{dx}{y}$. In particular, the element $\frac{dx}{y}$ from (2) is unique.

(4) Now compute the dimensions of $H^0(C, \Omega^1_C)$ and $H^1(C, \Omega^1_C)$. (Hint: the second number is a constant regardless of $P(x)$, whereas the first number depends solely on the degree of $P(x)$.)

(5) What do you observe?

Alright, now it's time to discuss general (smooth "complete") curves instead of the hyperelliptic ones. But before that, one needs to familiar oneself with some commutative algebra:

Exercise 1.5. (1) Find the notion of "finitely generated field extension" and "transcendental degree" of such a field extension.

(2) Learn about "valuation rings" and "valuations", focus on those that have prefix adjective "discrete".

(3) Find the notion of "taking integral closure of a ring in a field", focus on the finiteness property this process has.

(4) Familiarize yourself with the commutative algebra concerning the notion of "Dedekind domain".

Definition 1.6. Let k be a field, by a *function field of curve* we mean a finitely generated field extension K/k of transcendental degree 1.

Warning 1.7. Usually people will add some slightly technical conditions such as k is algebraically closed in K, or k is itself algebraically closed, or $K \otimes_k \bar{k}$ is still a field. Feel free to assume these conditions, and we will have to summon them when discussing Riemann–Roch and Serre duality.

Let's see if you have done the previous exercise.

Exercise 1.8. Let K/k be a function field of curve.

(1) Take an element $t \in K \setminus k$, what is the possibility of the sub-algebra $k[t] \subset K$? Similarly what is the possibility of the sub-field inside K generated by k and t?

(2) Convince yourself that very often in the situation of (1), K is a finite extension of the sub-field inside K generated by k and t. (Hint: this is exactly when t is a transcendental element for the extension K/k.)

(3) Show that any valuation ring squeezed between k and K: $k \subsetneq \mathcal{O}_v \subset K$ must be Noetherian. In fact, show that such an $\mathcal{O}_v$ is either the whole of K (so a field) or a DVR (short-hand for discrete valuation ring).

(4) Let $t \in K$ be a transcendental element, show that we have the following equality:

$$A := \text{integral closure of } k[t] \text{ in } K = \bigcap_{\text{valuation rings } k \subsetneq \mathcal{O}_v \subset K \text{ containing } t} \mathcal{O}_v.$$

(5) In the situation of (4), show that A is a Dedekind domain and is a finite flat $k[t]$-algebra. Furthermore, show that the set of valuation rings $\mathcal{O}_v$ containing t is in bijection with prime ideals of A, in fact $\mathcal{O}_v$'s containing t are just local rings at prime ideals of A. (Sanity check: the trivial valuation ring $\mathcal{O}_v = K$ corresponds to the only non-maximal prime ideal (0), and $K = Frac(A)$.)

There is a regular "complete" curve X_K functorially associated with a function field of curve K/k, constructed as follows.

Construction 1.9. Let K/k be a function field of curve. Consider the set of valuation rings $\{\mathcal{O}_v \mid k \subsetneq \mathcal{O}_v \subset K\}$, topologize it with an open basis given by subsets of the form $D(f_1, \ldots, f_n) := \{\mathcal{O}_v | f_i \in \mathcal{O}_v\}$ where $\{f_i\}$ is a finite subset in K. Define a sheaf of rings $\mathcal{O}$ whose section on an open U is a function $u \mapsto f(u) \in \mathcal{O}_u$ such that for any $u \in U$ there is a transcendental element $t \in \mathcal{O}_u$ (now temporarily denote the integral closure of $k[t]$ by $\overline{k[t]}$), and two elements $g, h \in \overline{k[t]}$ such that for all $u' \in U \cap D(t, h^{-1}) \subset D(t)$, we have $f(u') = g/h$ under the identification of $\mathcal{O}_{u'} = (\overline{k[t]})_{\mathfrak{p}'}$. (From the previous exercise, as apparently prime ideals of $\overline{k[t]}$ is identified with the set $D(t)$, under which the valuation rings and local rings also identifies.) The ringed space constructed is denoted by X_K.

Exercise 1.10. (1) Is the topology on X_K defined above the cofinite topology?

(2) Let t be a transcendental element in K, show that we have an isomorphism of ringed space $D(t) \cong Spec(\overline{k[t]})$. Since these $D(t)$'s cover X_K, we see that X_K is a scheme.

(3) Show that X_K is a regular 1-dimensional k-variety.

(4) Show that the local ring at the generic point of X_K is exactly K. (So in classical terms, K is the function field of the curve X_K.)

Definition 1.11. By a *curve* we will simply mean a k-variety X_K constructed above from a function field of curve K.

We can see some previous examples in this new framework.

Exercise 1.12. (1) Show that $\mathbb{P}^1_k$ is the curve associated with the function field $K = k(t)$.

(2) Show that the hyperelliptic curve constructed before is the curve associated with the function field $K = k(x)[y]/(y^2 - P(x))$.

2. Divisors and Line bundles

Let X_K be a curve associated with K/k.

Definition 2.1. A divisor on X_K is an element in the free abelian group generated by the set of closed points on X_K, said differently, a divisor is simply a formal sum $\sum_{x \in |X_K^{\mathrm{cl}}|} a_x x$ where x ranges over all closed points on X_K and coefficients $a_x \in \mathbb{Z}$ are 0 except for a finite number of x's.

They form the divisor group of X_K, $Div(X_K) \coloneqq \mathbb{Z}[\{\text{non-trivial valuations } v \text{ on } K/k\}]$.

A divisor is called *effective* if all of its coefficients $a_x \geq 0$. We write $D \geq 0$ to express that D is an effective divisor.

The *support* of a divisor is the finite set of x's whose coefficients $a_x \neq 0$.

One important class of divisors come from elements in $K^\times$.

Construction 2.2 (Principal divisors)**.** Given an element $f \in K^\times$, we define the associated divisor by $Div(f) \coloneqq \sum_v v(f) \cdot v$. Divisors arising this way are called *principal divisors*.

Exercise 2.3. (1) Check that this is indeed a finite sum.

(2) Show that we have an abelian group homomorphism: $K^\times \to Div(X_K)$, with kernel given by $\ell^\times$ where ℓ is the algebraic closure of k in K.

A very interesting notion in AG is that of "degree".

Definition 2.4 (Degree of points and divisors)**.** Given a closed point $x \in X_K$, define $\deg(x) \coloneqq [\kappa(x) \colon k]$. This extends uniquely to an abelian group homomorphism: $Div(X_K) \xrightarrow{\deg} \mathbb{Z}$, this gives us the notion of degree of a divisor.

Exercise 2.5 (finite map between curves)**.** Let L/K be an extension of function fields of curves, show that we have an induced map of associated curves $f \colon X_L \to Y_K$. Also show that the extension L/K necessarily has finite degree. Conversely, do you know how to characterize map of k-varieties $X_L \to Y_K$ which arises in this way?

Definition 2.6 (Degree of a finite map between curves)**.** We call a map $f \colon X_L \to Y_K$ a "finite map" if it arises in the above manner. For a finite map f, its degree is defined by $\deg(f) \coloneqq [L : K]$.

Construction 2.7 (Pullback of divisors)**.** Let $f \colon X_L \to Y_K$ be a finite map of curves corresponding to L/K, then given any point $y \in Y_K$ corresponds to a discrete valuation v on K, define an effective divisor on X_L by: $f^*(y) \coloneqq \sum_{w|v} e(w/v)w$, where w's ranging over all valuations of L extending that of K, and $e(w/v)$ is the ramification index. (Need to check that the above is a finite sum.) This extends uniquely to a map of divisors $f^* \colon Div(Y_K) \to Div(X_L)$.

Exercise 2.8. (1) Show that given a finite map of curves $f \colon X_L \to Y_K$, then for any divisor $D \in Div(Y_K)$, we have a formula: $\deg(f^*D) = \deg(f) \cdot \deg(D)$.

(2) Show that any transcendental $f \in K^\times$ defines a finite map $f \colon X_K \to \mathbb{P}^1_k$, corresponds to the finite field extension $k(t) \to K$ sending $t \mapsto f$. Show that we have a formula $f^*(Div(t)) = Div(f)$.

(3) Show that for any element $f \in K^\times$, one has that $\deg(Div(f)) = 0$. In particular, we have an abelian group homomorphism $K^\times \to Div^0(X_K) \subset Div(X_K)$, where $Div^0(X_K)$ denotes the subgroup of degree 0 divisors on X_K.

Is there any degree 0 divisor other than the principal ones? In other words, is the homomorphism above surjective? We'll discuss this below, but first let us briefly discuss vector bundles.

Exercise 2.9. Let X be a locally Noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X, show that TFAE:

- There is an open cover $\{U_i\}$ of X such that restriction of $\mathcal{F}$ to each U_i is of the form $\mathcal{O}_{U_i}^r$;
- For any point $x \in X$, the stalk is of the form $\mathcal{F}_x \simeq \mathcal{O}_{X,x}^{r(x)}$;
- For any affine open $U = Spec(A) \subset X$, the A-module $\mathcal{F}(U)$ is finitely generated locally free;
- For any affine open $U = Spec(A) \subset X$, the A-module $\mathcal{F}(U)$ is finitely generated projective;
- For any affine open $U = Spec(A) \subset X$, the A-module $\mathcal{F}(U)$ is finitely generated flat;

Note that in the above, the number $r(x)$ might vary, show that $r(x)$ is locally constant.

Definition 2.10. In the above setting, a coherent sheaf satisfying equivalent conditions above is called "locally free". If the function $r(x)$ above is constant r, we call it a vector bundle of rank r. A line bundle is a vector bundle of rank 1.

Construction 2.11. Many natural constructions in linear algebra can be extended to locally free sheaves. For instance:

- We may form direct sum $\oplus$ and tensor product of $\otimes$ of two locally free sheaves of ranks r and r', resulting another vector bundles of rank $r + r'$ and $r \cdot r'$.
- Given any locally free sheaf V, we can form its dual, given by $V^\vee := \underline{Hom}(V, \mathcal{O}_X)$.
- For any number m, we can similarly form m-th symmetric power, m-th exterior power, m-th divided power of a locally free sheaf. (Feel free to ignore this item if you don't know what I mean.)

Exercise 2.12. Make sense of the above. Then show that given two locally free sheaves V_1 and V_2, one has the following equality: $\mathrm{Hom}(V_1, V_2) = \Gamma(V_1^\vee \otimes V_2)$.

Definition 2.13. The set of line bundles on X_K, together with tensor product, forms an abelian group with $\mathcal{O}_{X_K}$ being the identity element. This abelian group is called the Picard group of X_K, denoted by $Pic(X_K)$.

Construction 2.14 (Divisor to Line bundle)**.** Let X_K be a curve associated with K/k. Let $D = \sum_x a_x \cdot x \in Div(X_K)$, define a line bundle by $\mathcal{O}(D)(U) := \{f \in K | \, (Div(f) + D)\,|_U \geq 0\}$ for any non-empty $U \subset X_K$.

Exercise 2.15. (1) Check that $\mathcal{O}(D)$ is indeed a line bundle by giving it a different description, using Dedekind factorization.

(2) Show that the construction $D \mapsto \mathcal{O}(D)$ gives rise to an abelian group homomorphism $Div(X_K) \to Pic(X_K)$.

(3) Take any $f \in K^\times$ and $D \in Div(X_K)$, show that "divide by f" defines an isomorphism $\mathcal{O}(D) \xrightarrow[\simeq]{-/f} \mathcal{O}(D + Div(f))$.

(4) Show that when $\deg(D) < 0$, the associated line bundle has only zero global section $\mathcal{O}(D)(X_K) = \{0\}$.

(4') Show that when $\deg(D) = 0$, then either D is non-principal and the associated line bundle has again only zero global section; or D is principal, in which case the associated line bundle is just $\mathcal{O}(D) \simeq \mathcal{O}_{X_K}$.

Notation 2.16. If $\mathcal{L}$ is a line bundle and $\mathcal{F}$ is a quasi-coherent sheaf, then we may form another quasi-coherent sheaf $\mathcal{F} \otimes \mathcal{L}$. When $\mathcal{L} = \mathcal{O}(D)$, we simply denote $\mathcal{F} \otimes \mathcal{O}(D)$ by $\mathcal{F}(D)$. One may interpret sections of $\mathcal{F}(D)$ as "rational/meromorphic" sections of $\mathcal{F}$ which has poles/zeros bounded below by $-D$", we will make this more precise later.

Theorem 2.17. *For any function field of curve K/k, we have an exact sequence: $1 \to \ell^\times \to K^\times \to Div(X_K) \to Pic(X_K) \to 1$, where ℓ is the algebraic closure of k in K.*

The content is to show exactness at Div and Pic, we deal with Div first. We need one more construction/jargon:

Construction 2.18 (Structure sheaf of effective divisors)**.** Let $D = \sum_x a_x \cdot x \geq 0$ be an effective divisor on X_K, its "structure sheaf" is defined to be $\mathcal{O}_D := \prod_{x \in supp(D)} \mathcal{O}_X / \mathfrak{m}_x^{a_x}$.

Exercise 2.19. (1) Check that $\mathcal{O}_D$ is a coherent sheaf, and there is a natural exact sequence of coherent sheaves:

$$0 \to \mathcal{O}(-D) \cong \mathcal{O}(D)^\vee \to \mathcal{O}_{X_K} \to \mathcal{O}_D \to 0.$$

Here the map $\mathcal{O}(-D) \to \mathcal{O}_{X_K}$ is the same as an element in $\mathcal{O}(D)(X_K)$ (by the exercise about duality of vector bundles), and corresponds to the element $1 \in K$ (since $Div(1) + D = D \geq 0$, we may regard 1 as an element in $\mathcal{O}(D)(X_K)$).

(2) Now suppose D is a divisor such that $\mathcal{O}(D)$ is the trivial line bundle. Then choose an isomorphism $\alpha\colon \mathcal{O}_{X_K} \to \mathcal{O}(D)$, so the global section 1 on the left hand side will correspond to the right hand side an element $f \in K$ such that $Div(f) + D \geq 0$. Now using (1), show that the cokernel of α is precisely $\mathcal{O}_{Div(f)+D}$.

(3) Therefore the vanishing of cokernel of α is exactly $Div(f) + D = 0$, in other words, $D = Div(f^{-1})$ is principal.

Lastly we need to show that every line bundle on X_K is of the form $\mathcal{O}(D)$ for some divisor D. What we need is the following jargon:

Definition 2.20. Let $\mathcal{L}$ be a line bundle on X_K, a rational trivialization of $\mathcal{L}$ is an isomorphism $\alpha\colon K \xrightarrow{\simeq} \mathcal{L}_\eta$ between the function field and the stalk as K-vector spaces.

Remark 2.21. The difference between K and $\mathcal{L}_\eta$ is that the former is equipped with a canonical base element $1 \in K$ whereas the latter is only a 1-dimensional K-vector space without a canonical choice of base.

Exercise 2.22. (1) Show that rational trivializations always exist.

(2) In fact, show that rational trivializations of $\mathcal{L}$ is in bijection with nonzero elements in $\mathcal{L}_\eta$.

(3) Find the definition of "torsor" and show that the set of rational trivializations of any line bundle $\mathcal{L}$ is a $K^\times$-torsor. (Feel free to ignore this item if you don't want to think about these abstract stuff for now.)

Next we need to know that rational trivializations can be extended to a trivialization on a non-empty open:

Exercise 2.23. Let $\mathcal{L}$ be a line bundle on X_K with a rational trivialization $\alpha\colon K \xrightarrow{\simeq} \mathcal{L}_\eta$. Show that there is a non-empty open $U \subset X_K$ and an isomorphism $\alpha_U\colon \mathcal{O}_U \xrightarrow{\simeq} \mathcal{L}|_U$ such that its restriction to the stalk at η is α.

Recall that $X_K \setminus U$ is a finite set of closed points on X_K, we want to find an appropriate divisor D supported on this finite set so that the isomorphism α_U extends to an isomorphism $\alpha_X\colon \mathcal{O}(D) \xrightarrow{\simeq} \mathcal{L}$. To this end, we first study the question of the necessary and sufficient conditions on D so that α_U can be extended to a morphism at all. The following "local" discussion is the key:

Definition 2.24. Let $\mathcal{O}$ be a DVR with fraction field K and uniformizer t. Given a datum of a triple (M, N, α_K) where M and N are rank 1 finite free $\mathcal{O}$-modules and $\alpha_K\colon M_K \xrightarrow{\simeq} N_K$ is an isomorphism. Then we define $ord(N/M) \coloneqq \min\{n \in \mathbb{Z} \mid t^{-n} \cdot \alpha_K(M) \supset N \text{ as submodules in } N_K\}$.

Remark 2.25. In modern language, M and N are called rank 1 lattices in the vector space $M_K \cong N_K$, and $ord(M/N)$ is measuring the difference between these two lattices. Obviously $ord(N/M) \geq 0$ if and only if $M \subset N$, and when this happens $ord(N/M)$ is the length of the $\mathcal{O}$-module N/M. A warning is that contrary to what the notation suggests, this quantity $ord(N/M)$ depends crucially on the isomorphism α_K, in fact one might say that this quantity depends only on α_K.

Now suppose we have two line bundles $\mathcal{L}_i$, a non-empty open $U \subset X_K$ and an isomorphism $\alpha_U\colon (\mathcal{L}_1)|_U \to (\mathcal{L}_2)_U$, we may use the above "local" definition to define a "global" divisor:

Definition 2.26. In the above setting, we define $D(\mathcal{L}_2/\mathcal{L}_1)$ by:

$$D(\mathcal{L}_2/\mathcal{L}_1) \coloneqq \sum_x ord(\mathcal{L}_{2,x}/\mathcal{L}_{1,x}) \cdot x.$$

Here we are using the fact that local ring $\mathcal{O}_{X,x}$ is a DVR, and these stalks are equipped with a rational identification $\alpha_U|_\eta$, so the *ord* between these two stalks make sense.

Exercise 2.27. (1) Show that the isomorphism α_U extends to a morphism $\mathcal{L}_1 \otimes \mathcal{O}(D) \to \mathcal{L}_2$ if and only if $D \leq D(\mathcal{L}_2/\mathcal{L}_1)$.

(2) Show that the isomorphism α_U extends to an isomorphism $\mathcal{L}_1 \otimes \mathcal{O}(D(\mathcal{L}_2/\mathcal{L}_1)) \xrightarrow{\simeq} \mathcal{L}_2$.

(3) In particular, combine what we have done so far and conclude that every line bundle on X_K is of the form $\mathcal{O}(D)$ for some divisor D.

It's okay to be puzzled by the above at this point, please try to look at the discussion starting at Definition 3.16 and afterwards, I believe that can clarify things quite a bit.

Definition 2.28. For any line bundle $\mathcal{L}$ on X_K, define its degree by $\deg(\mathcal{L}) := \deg(D)$ for any D such that $\mathcal{L} \simeq \mathcal{O}(D)$.

By the theorem we have just proved, two different such D's differ by a principal divisor, hence will have the same degree, therefore the above definition makes sense. Our earlier quest of finding a degree zero but non-principal divisor now translates to finding a degree zero but nontrivial line bundle. Here's a somewhat difficult exercise:

Exercise 2.29. For each degree $d \geq 3$, show that there are examples of hyperelliptic curves X_K associated with degree d polynomials $P(x) \in k[x]$, such that there are degree zero but nontrivial line bundles on X_K.

3. Cohomology of curves

In this section, we will discuss various results concerning cohomology of curves.

3.1. **finiteness.** For starter, let's discuss finiteness results, again we fix a field k. Recall that X_K's are quasi-compact and covered by *Spec* of Dedekind domains, so they are Noetherian schemes. As such, we can talk about coherent sheaves, which are just quasi-coherent sheaves whose restriction to each affine open comes from finitely generated modules. Here's the finiteness result that we aim for in this subsection.

Theorem 3.1. *Let X_K be a curve.*

Easy vanishing: For any quasi-coherent sheaf $\mathcal{G}$ on X_K, we have $H^{\geq 2}(X_K, \mathcal{G}) = 0$.

Finiteness: Let $\mathcal{F}$ be a coherent sheaf on X_K, then $H^0(X_K, \mathcal{F})$ and $H^1(X_K, \mathcal{F})$ are finite dimensional k-vector spaces.

We shall prove this by studying the case of $\mathbb{P}^1_k$ carefully. Recall that $\mathbb{P}^1_k := Proj(k[X,Y])$ with $|X| = |Y| = 1$. However, here's an easier way of looking at it.

Exercise 3.2. (1) Show that $Proj(k[X,Y])$ is covered by $D(X)$ and $D(Y)$.

(2) Show that $D(X) \cong Spec(k[Y/X])$ whereas $D(Y) \cong Spec(k[X/Y])$. Show that $D(X) \cap D(Y) \cong Spec(k[Y/X, X/Y])$.

(3) Prove the easy vanishing part of Theorem 3.1 for $\mathbb{P}^1_k$.

So we see that $\mathbb{P}^1_k$ is glued from $Spec(k[t])$ and $Spec(k[t^{-1}])$ along their common open $Spec(k[t^{\pm 1}])$, where t and t^{-1} are symbols representing Y/X and X/Y. From this description, we can understand (quasi-)coherent sheaves on it quite explicitly.

Exercise 3.3. (1) Show that the category of quasi-coherent (resp. coherent) sheaves on $\mathbb{P}^1_k$ is the same as the category of triples (M, N, α) where M and N are (resp. finitely generated) $k[t]$- and $k[t^{-1}]$-modules, and $\alpha\colon M[1/t] \xrightarrow{\simeq} N[t]$ is an isomorphism of $k[t^{\pm 1}]$-modules.

(1') Show that if M is t-torsion free, and N is t^{-1}-torsion free, we may regard the triple as the following diagram: $M \subset M[t^{-1}] = N[t] \supset N$.

(2) Recall that we have constructed certain coherent sheaves $\mathcal{O}(n) := \widetilde{S(n)}$ on $Proj(S)$. Under the identification of (1) or (1'), what triple does $\mathcal{O}(n)$ corresponds to?

(3) Show that if a quasi-coherent sheaf $\mathcal{G}$ corresponds to (M, N, α). Then to give a map $\mathcal{O}(d) \to \mathcal{G}$ is the same as giving a pair of elements $(m, n) \in M \times N$ such that $t^d \alpha(m) = n$ in $M[t^{-1}] \xrightarrow[\alpha]{\simeq} N[t]$.

(4) Show that if a quasi-coherent sheaf $\mathcal{G}$ corresponds to (M, N, α). Pick any element $m \in M$, then for $d \gg 0$ one can find the n in (3) to define a map $\mathcal{O}(-d) \to \mathcal{G}$.

Definition 3.4. Let $\mathcal{F}$ be a coherent sheaf on a curve X_K, its *generic rank* is defined to be $\dim_K \mathcal{F}_\eta$ where η is the generic point of X_K whose local ring was showed to be the corresponding function field of curve K.

Exercise 3.5. Convince yourself that the above definition can be generalized to coherent sheaves on integral locally Noetherian schemes.

Now we are ready to prove the finiteness part of Theorem 3.1 for $\mathbb{P}^1_k$.

Exercise 3.6. Prove the finiteness part of Theorem 3.1 for $\mathbb{P}^1_k$ by induction on generic rank of the coherent sheaf $\mathcal{F}$, here's some steps.

(1) If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^1_k$ having generic rank 0, show that $H^0(\mathbb{P}^1_k, \mathcal{F})$ is finite dimensional and $H^1(\mathbb{P}^1_k, \mathcal{F}) = 0$.

(2) If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^1_k$ having generic rank $r > 0$, show that there exists some $d \gg 0$ and an injection of sheaves $\mathcal{O}(-d) \hookrightarrow \mathcal{F}$ on $\mathbb{P}^1_k$. In particular, we have a short exact sequences of coherent sheaves:

$$0 \to \mathcal{O}(-d) \to \mathcal{F} \to \mathcal{Q} \to 0.$$

(3) In situation of (2), show that $\mathcal{Q}$ has generic rank 1 less than that of $\mathcal{F}$, hence by induction we know finiteness of cohomology of $\mathcal{Q}$.

(4) Use "long exact sequence" and finiteness of cohomology of $\mathcal{O}(-d)$ (please explicitly compute to prove this) to conclude the finiteness of cohomology of $\mathcal{F}$.

In fact, the case of $\mathbb{P}^1_k$ implies the general case, by the following trick.

Exercise 3.7. Let X_K be the curve associated with K, let $t \in K$ be a transcendental element.

(1) Recall that we have identifications of open subschemes in X_K: $D(t) \cong \mathit{Spec}(\overline{k[t]})$ and $D(t^{-1}) \cong \mathit{Spec}(\overline{k[t^{-1}]})$.

(2) Show that $D(t)$ and $D(t^{-1})$ covers X_K with their intersection

$$D(t) \cap D(t^{-1}) = D(t, t^{-1}) = \mathit{Spec}(\overline{k[t]}[t^{-1}]) = \mathit{Spec}(\overline{k[t^{-1}]}[t]).$$

(3) Conclude that there is a map $\pi\colon X_K \to \mathbb{P}^1_k$ which on the above affine pieces are given by finite algebra map. (Notice that here we said finite instead of merely finite type.)

(4) Given a coherent sheaf $\mathcal{F}$ on X_K, its restriction to both $D(t)$ and $D(t^{-1})$ are given by finite modules M and N over $\overline{k[t]}$ and $\overline{k[t^{-1}]}$. Show that we may regard them as finite modules over $k[t]$ and $k[t^{-1}]$, and they "glue" to a coherent sheaf on $\mathbb{P}^1_k$, denoted by $\pi_*\mathcal{F}$.

(5) Show that we have natural identifications $H^*(X_K, \mathcal{F}) = H^*(\mathbb{P}^1_k, \pi_*\mathcal{F})$. This concludes our proof of Theorem 3.1.

Using this finiteness, we can make the following definition.

Definition 3.8. Let X_K be a curve such that k is algebraically closed in K, then its arithmetic genus is defined to be $g \coloneqq \dim_k H^1(X_K, \mathcal{O})$.

Exercise 3.9. (1) When K is the function field of $\mathbb{P}^1_k$ or a hyperelliptic curve associated with $P(x)$, show that k is algebraically closed in K.

(2) Compute the arithmetic genus g for those mentioned in (1).

3.2. **Riemann–Roch I: an a priori estimate.** From now on, we shall assume that k is an algebraically closed field. As usual, we still denote a general function field of curve by K/k. By a general fact stated in the previous part, we know that X_K are smooth over k: for varieties over perfect fields, regular is equivalent to smooth.

Now we start investigating cohomology of line bundles on curves, the two most important results are Riemann–Roch's theorem and Serre duality. We need the following:

Definition 3.10. Let $\mathcal{F}$ be a coherent sheaf on X_K, then its Euler characteristic is defined to be $\chi(X_K, \mathcal{F}) \coloneqq \dim_k H^0(X_K, \mathcal{F}) - \dim_k H^1(X_K, \mathcal{F})$. We shall simplify the notation by writing only $\chi(\mathcal{F})$. People often denote $\dim_k H^i$ by h^i.

Here's one simple fact:

Exercise 3.11 (Euler characteristic is additive in short exact sequences)**.** Show that if we have a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

then $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$.

The theorem of Riemann–Roch is now very simple:

Theorem 3.12. *The following constant $\chi(\mathcal{L}) - \deg(\mathcal{L})$ doesn't depend on the line bundle $\mathcal{L}$ on X_K.*

Exercise 3.13. Prove the above theorem, using the following idea:

(1) First of all the line bundle must be of the form $\mathcal{L} \simeq \mathcal{O}(D)$ for some divisor $D = D_1 - D_2$ with $D_i \geq 0$ effective.

(2) Then exploit the following two short exact sequences:

$$0 \to \mathcal{O} \to \mathcal{O}(D_1) \to \mathcal{O}(D_1) \otimes_{\mathcal{O}} \mathcal{O}_{D_1} \simeq \mathcal{O}_{D_1} \to 0;$$

and

$$0 \to \mathcal{O}(D_1 - D_2) \to \mathcal{O}(D_1) \to \mathcal{O}(D_1) \otimes_{\mathcal{O}} \mathcal{O}_{D_2} \simeq \mathcal{O}_{D_2} \to 0,$$

to show that $\chi(\mathcal{L}) - \deg(\mathcal{L}) = \chi(\mathcal{O}) - \deg(\mathcal{O})$.

Recall that the arithmetic genus was defined to be $h^1(\mathcal{O})$. Let us unfold the above result.

Exercise 3.14. (1) Show that $\chi(\mathcal{L}) = 1 - g + \deg(\mathcal{L})$.

(2) Consequently, show that $h^0(\mathcal{L}) \geq 1 - g + \deg(\mathcal{L})$.

Remark 3.15. The above inequality is what Riemann discovered, of course in a slightly different language/setting, and Roch (a student of Riemann) found the error term.

To prepare us for what's gonna happen later, let us understand $H^1(X_K, \mathcal{L})$ better.

Definition 3.16. Let $\mathcal{L}$ be a line bundle on X_K, the (quasi-coherent) sheaf of rational sections $j_*\mathcal{L}_\eta$ of $\mathcal{L}$ is defined as following: for any non-empty $U \subset X_K$, we have $j_*\mathcal{L}_\eta(U) := \mathcal{L}_\eta$ (which is nothing but a 1-dimensional K-vector space); and for empty U, the section is just $\{0\}$. Restriction maps are either identity or zero map.

Exercise 3.17. (1) Show that the above defines a sheaf, in fact, show that the above defines a quasi-coherent sheaf on X_K.

(2) Show that there is a natural injection of quasi-coherent sheaves: $\mathcal{L} \hookrightarrow j_*\mathcal{L}_\eta$.

(3) How would you describe the cokernel of the above injection?

(4) Show that for any Dedekind domain A, the quotient $Frac(A)/A$ as an A-module can be alternatively described as: $Frac(A)/A \simeq \bigoplus_{(0)\neq\mathfrak{p}\subset A} Frac(A)/A_{\mathfrak{p}}$.

(5) Really, how would you describe the cokernel of the above injection?

(6) Show that there is a natural short exact sequence associated with any line bundle $\mathcal{L}$ on X_K as follows:

$$0 \to \mathcal{L} \to j_*\mathcal{L}_\eta \to \bigoplus_{x \in X_K^{cl}} \mathcal{L}_\eta/\mathcal{L}_x \to 0.$$

Convince yourself that the above is a SES of quasi-coherent sheaves. Here X_K^{cl} simply denotes the closed points of X_K.

With the above natural short exact sequence, we can better understood the final step of the proof of surjectivity of $Div \to Pic$: we were simply comparing the two natural SES's associated with $\mathcal{L}$ and $\mathcal{O}$. The rational isomorphism pre-fixed yields an isomorphism of the middle term, so we may think of both $\mathcal{L}$ and $\mathcal{O}$ as subsheaves inside a common quasi-coherent sheaf. Now the question of whether $\mathcal{L}$ contains $\mathcal{O}$ (both viewed as subsheaves of this common quasi-coherent sheaf) becomes whether the quotient

$$\mathcal{O}_\eta = K \xrightarrow[\simeq]{\alpha} \mathcal{L}_\eta \twoheadrightarrow \bigoplus_{x \in X_K^{cl}} \mathcal{L}_\eta/\mathcal{L}_x$$

factors through the quotient

$$\mathcal{O}_\eta \twoheadrightarrow \bigoplus_{x \in X_K^{cl}} \mathcal{O}_\eta/\mathcal{O}_x.$$

The fact that the rational isomorphism extends to an isomorphism on an open part shows that for almost all $x \in X_K^{cl}$, the lattices $\mathcal{L}_x$ and $\mathcal{O}_x$ inside $\mathcal{O}_\eta = K \xrightarrow[\simeq]{\alpha} \mathcal{L}_\eta$ are identified. So we only have to worry about finitely many points in the complement of U, then the divisor $D = ord(\mathcal{L}/\mathcal{O})$ is exactly designed to match the lattices appropriately, so that the subsheaves $\mathcal{O}(D)$ and $\mathcal{L}$ are exactly identified.

Exercise 3.18. Understand the above mumbling. The following might be helpful: let's say $\mathcal{L}$ is a line bundle on X_K, and let's say $D \geq 0$ is an effective divisor. Can you write down some natural diagrams comparing the natural SES's for $\mathcal{L}$, $\mathcal{L} \otimes \mathcal{O}(D)$, and $\mathcal{L} \otimes \mathcal{O}(-D)$.

Finally, let's give a better understanding of $H^1(\mathcal{L})$.

Exercise 3.19. Let $\mathcal{L}$ be a line bundle on X_K.

(1) Show that $H^{\geq 1}(j_*\mathcal{L}_\eta) = 0$ and that $H^{\geq 1}(\bigoplus_{x \in X_K^{cl}} \mathcal{L}_\eta/\mathcal{L}_x) = 0$.

(2) How would you interpret $H^0(\bigoplus_{x \in X_K^{cl}} \mathcal{L}_\eta/\mathcal{L}_x)$?

(3) Describe the above H^0, you may find the following phrase helpful: Laurent tails of "rational/meromorphic sections" of $\mathcal{L}$ at finitely many points on X_K.

(4) Then interpret $H^1(\mathcal{L})$ as:"it's precisely the failure of finding rational sections of $\mathcal{L}$ matching prescribed Laurent tails of 'rational/meromorphic sections' of $\mathcal{L}$."

Let us explicate the above when $\mathcal{L} = \mathcal{O}(D)$ for some divisor D.

Notation 3.20. If D is a divisor, we denote $\mathcal{T}[D] := \bigoplus_{x \in X_K^{cl}} K/\mathcal{O}(D_1)_x$.

Exercise 3.21. Check that $H^1(\mathcal{O}(D)) \cong \mathrm{Coker}(K \to \mathcal{T}[D])$.

Now we shall introduce two kinds of operators between $H^1(\mathcal{O}(D))$ for various D's.

Construction 3.22. (1) When $D_1 \leq D_2$, then we have a natural map $\mathcal{O}(D_1) \to \mathcal{O}(D_2)$. This induces the following commutative diagram between SES's:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(D_1) & \longrightarrow & j_*K & \longrightarrow & \mathcal{T}[D_1] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(D_2) & \longrightarrow & j_*K & \longrightarrow & \mathcal{T}[D_2] & \longrightarrow & 0. \end{array}$$

In particular, the induced map $H^1(\mathcal{O}(D_1)) \to H^1(\mathcal{O}(D_2))$ comes from a map $t_{D_2}^{D_1} \colon \mathcal{T}[D_1] \to \mathcal{T}[D_2]$, which can be interpreted as "truncating prescribed Laurent tails".

(2) Recall that given $f \in K^\times$, one gets a natural map $\mathcal{O}(D) \xrightarrow{\cdot f} \mathcal{O}(D - Div(f))$. The induced map $H^1(\mathcal{O}(D)) \to H^1(\mathcal{O}(D - Div(f)))$ comes from a map $\mu_f \colon \mathcal{T}[D] \to \mathcal{T}[D - Div(f)]$, which can be interpreted as "multiplying prescribed Laurent tails by appropriate tails of f".

Exercise 3.23. Understand what $t_{D_2}^{D_1}$ and μ_f is doing, it can be helpful to think about it one point at a time. (So in class I should probably talk about the "local/DVR version" of these maps on tails.)

Here is a complicated looking problem: let D be a divisor, and let C be an effective divisor. Consider the following construction: given any $f \in H^0(\mathcal{O}(C))$, then we can define a composition

$$\mathcal{T}[D - C] \xrightarrow{\mu_f} \mathcal{T}[D - C - Div(f)] \xrightarrow{t_D^{D-C-Div(f)}} \mathcal{T}[D].$$

Exercise 3.24. (1) Check that the above composition is well-defined. (The potential issue is that when defining this t operator, you need an inequality between divisors, and in our situation it's fine, why?)

(2) Show that $f \mapsto t \circ \mu_f$ defines a *linear*(!) map

$$H^0(\mathcal{O}(C)) \to \mathrm{Hom}(\mathcal{T}[D - C], \mathcal{T}[D]).$$

3.3. **Riemann–Roch II: Serre duality.** Strictly speaking, Serre duality is an independent statement. The reason why we put it under the umbrella of Riemann–Roch is really because these two only yield the strongest power when coupled together. Also, our proof for Serre duality will actually use the a priori estimate coming from the previous subsection.

First of all, let us state the theorem.

Theorem 3.25 (Serre duality)**.** *For any divisor D on X_K, there is a natural bi-linear pairing $H^0(X_K, \Omega^1(-D)) \times H^1(X_K, \mathcal{O}(D)) \xrightarrow{res} k$ inducing an isomorphism $H^0(X_K, \Omega^1(-D)) \xrightarrow{\cong} H^1(X_K, \mathcal{O}(D))^\vee$. In particular we have $h^1(\mathcal{O}(D)) = h^0(\Omega^1(-D))$.*

Recall that $\Omega^1_{X_K}$ denotes the sheaf of 1-forms on X_K, which is a line bundle on X_K. Here is a concrete way to understand an element in $H^0(\Omega^1(D))$.

Let $x \in X_K^{cl}$ which is in 1-1 correspondence with discrete valuations v_x on K/k, let t be a uniformizer in $\mathcal{O}_x$ (for a DVR, a uniformizer is just an element in the maximal ideal which is not in the square of the maximal ideal, aka an element with $v_x(t) = 1$). Then $\Omega^1_x \simeq \mathcal{O}_x \cdot dt$ and $\Omega^1_K \simeq K \cdot dt$. So an element in $H^0(\Omega^1(D))$ is nothing but a rational 1-form $\omega \in \Omega^1_K \cong \Omega^1_{K/k}$ such that for every point $x \in X_K^{cl}$, if we write $\omega = f \cdot dt$, then $f \in \mathcal{O}(D)_x$: this exactly means that $v_x(f) \geq -a_x$ where a_x is the coefficient of x in D.

Residue Theorem. Below let us carefully define the pairing.

Exercise 3.26. (1) Show that there is a decomposition which depends on choice of t: $K = (\mathfrak{m}_x)^b \bigoplus_{i<b} k \cdot t^i$.

(2) Show that if we are given an element $\omega \in \Omega^1_K$ and write $\omega = f \cdot dt$, then we can uniquely decompose $f = \sum_{i<0} a_i t^i + g$ with $a_i \in k$ and $g \in \mathcal{O}_x$. In other words, any element $\overline{\omega} \in \Omega^1_K / \Omega^1_x$ can be uniquely written as $\overline{\omega} = \sum_{i<0} a_i t^i \cdot dt$ with $a_i \in k$.

Here are two difficult facts that we shall summon:

Fact 3.27 (Residue Theorem)**.**

(1) A priori the coefficients of $\overline{\omega}$ above depend on the choice of t. But actually the coefficient $a_{-1} \in k$ is independent of the choice of t! Therefore we get a k-linear functional

$$res_x = a_1 \colon \Omega^1_K / \Omega^1_x \to k.$$

(2) For any nonzero $\omega \in \Omega^1_K$, it's easy to see that $res_x(\omega) = 0$ except for finitely many points $x \in X_K^{cl}$. What's difficult to see is that the following sum vanishes $\sum_{x \in X_K^{cl}} res_x(\omega) = 0$!

Here is something which I don't know how to do "by hand":

Exercise 3.28. Prove the fact (1) above.

We will prove the above two statements for curves over complex numbers. In Tate's paper "Residues of differentials on curves" readers can find elegant proofs of the above two facts.

Residue Pairing. So here's the pairing using the Residue theorem above.

Exercise 3.29. (1) Show that we have a natural map $\mathcal{T}[D] \otimes_k H^0(\Omega^1(-D)) \to \bigoplus_{x \in X_K^{cl}} \Omega^1_K / \Omega^1_x$.

Concretely, at a closed point x whose coefficient in D is a_x, the first tensor factor is $K/(\mathfrak{m}_x)^{-a_x}$, and the second tensor factor gives rise to an element in $\mathfrak{m}_x^{a_x} \otimes_{\mathcal{O}_x} \Omega^1_K$. So we may tensor them up and map to an element in $\Omega^1_K / \Omega^1_x = (K/\mathcal{O}_x) \otimes_{\mathcal{O}_x} \Omega^1_K$.

(2) Explicate the above "local analysis" around x in terms of a uniformizer $t \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$.

(3) Show that if a Laurent tail in $\mathcal{T}[D]$ comes from a rational function f, then it pairs with any rational 1-form $\omega \in H^0(\Omega^1(-D))$ will come from an element in Ω^1_K (which of course is none other than $f\omega$).

(4) Consequently, show that the following composition:

$$\mathcal{T}[D] \otimes_k H^0(\Omega^1(-D)) \to \bigoplus_{x \in X_K^{cl}} \Omega^1_K / \Omega^1_x \xrightarrow{res = \sum_x res_x} k$$

descends to a linear map

$$H^1(X_K, \mathcal{O}(D)) \otimes_k H^0(\Omega^1(-D)) \to k.$$

The above is the map appeared in Serre duality.

Exercise 3.30. Specialize to the case where $\mathcal{O}(D) \simeq \Omega^1$, what concrete statement can we deduce?

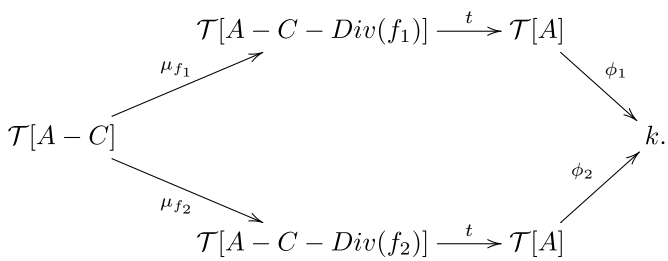
Proof of Serre duality. We look at the induced map $H^0(X_K, \Omega^1(-D)) \xrightarrow{\cong} H^1(X_K, \mathcal{O}(D))^\vee$, which we will denote by $\omega \mapsto Res_\omega$ (and interpret the latter symbol as the linear functional spelled out above). We need to show injectivity and surjectivity. Here are two easy lemmas.

Exercise 3.31. (1) Show that the above map is injective.

(2) Suppose that $D_2 \geq D_1$ and $\omega \in H^0(\Omega^1(-D_1))$. If Res_ω vanishes on the kernel of $H^1(\mathcal{O}(D_1) \to H^1(\mathcal{O}(D_2))$, then $\omega \in H^0(\Omega^1(-D_2)) \subset H^0(\Omega^1(-D_1))$ and the descended linear functional on $H^1(\mathcal{O}(D_2))$ is again given by Res_ω.

Here is a key lemma.

Lemma 3.32. *Let A be a divisor, and let ϕ_1 and ϕ_2 be two linear functionals on $H^1(\mathcal{O}(A))$. Then there exists a positive divisor C and $f_1,\ f_2 \in H^0(\mathcal{O}(C))$. Such that the following is a commutative diagram:*



If you prefer an equation, here is one:

$$\phi_1 \circ t_A^{A-C-Div(f_1)} \circ \mu_{f_1} = \phi_2 \circ t_A^{A-C-Div(f_2)} \circ \mu_{f_2},$$

as linear functionals on $H^1(\mathcal{O}(A - C))$.

Proof. For every positive divisor C, we have a linear map

$$H^0(\mathcal{O}(C)) \oplus H^0(\mathcal{O}(C)) \to \big(H^1(\mathcal{O}(A - C))\big)^\vee$$

defined by sending a pair (f_1, f_2) to

$$\phi_1 \circ t_A^{A-C-Div(f_1)} \circ \mu_{f_1} - \phi_2 \circ t_A^{A-C-Div(f_2)} \circ \mu_{f_2}.$$

(Recall that in a previous exercise, we have checked that $t \circ \mu_f$ is indeed linear!)

We need to show that there is some positive C such that the map above has nonzero kernel. It suffices to show that when C has degree large enough, the above source will have dimension larger than the target. What a great exercise?! □

Exercise 3.33. Finish the above proof. More precisely, using the Riemann–Roch theorem from previous subsection to show that: when $\deg(C) = d$ goes to infinity, the dimension of the source grows at least $2d$, whereas the dimension of the target grows at most d (up to a constant depending on A and the genus of X_K).

Now we are ready to prove Serre duality. Here is our last preparation.

Exercise 3.34. (1) If $D_1 \leq D_2$ and $\omega \in H^0(\Omega^1(-D_2)) \subset H^0(\Omega^1(-D_1))$, then show that $Res_\omega \circ t_{D_2}^{D_1} = Res_\omega$ as linear functionals on $H^1(\mathcal{O}(D_1))$.

(2) If D is a divisor and $\omega \in H^0(\Omega^1(-D))$, and let $f \in K^\times$. Then show: $Res_\omega \circ \mu_f = Res_{f\omega}$ as linear functionals on $H^1(\mathcal{O}(D + Div(f)))$.

Finally, let's prove Serre duality. So we are trying to show that every linear functional $\phi\colon H^1(\mathcal{O}(D)) \to k$ is of the form $Res_{\omega'}$ for some $\omega' \in H^0(\Omega^1(-D))$. What if we just take some nonzero $\omega \in \Omega^1_K$ and try to massage it into the desired ω'? Here are the steps that you shall carry out.

Exercise 3.35. (1) With notations as above, show that there exists a divisor A with $A \leq D$ and $\omega \in H^0(\Omega^1(-A))$.

(2) Let $\phi_1 = \phi \circ t_D^A \colon H^1(\mathcal{O}(A)) \to k$, and let $\phi_2 = Res_\omega$, then apply Lemma 3.32 to see that we are given ourselves some divisor C and two $f_i \in H^0(\mathcal{O}(C))$ with

$$\phi \circ t_D^A \circ t_A^{A-C-Div(f_1)} \circ \mu_{f_1} = Res_\omega \circ t_A^{A-C-Div(f_2)} \circ \mu_{f_2},$$

as linear functional on $H^1(\mathcal{O}(A-C))$.

(3) Show that the RHS of the above is just $Res_{f_2\omega}$.

(3') Show that we get an equality

$$\phi \circ t_D^{A-C-Div(f_1)} = Res_{(f_2/f_1)\omega}$$

as linear functional on $H^1(\mathcal{O}(A-C-Div(f_1))$.

(4) Since the RHS, as a linear functional, vanishes on the kernel of $t_D^{A-C-Div(f_1)}$, we see that $(f_2/f_1)\omega \in H^0(\Omega^1(-D))$, and the above equation just becomes $\phi = Res_{(f_2/f_1)\omega}$, we win!

3.4. **Quick applications.** Let's harvest. Recall that arithmetic genus was defined as $g = h^1(\mathcal{O})$.

Exercise 3.36. Show that $\dim_k H^0(\Omega^1_{X_K}) = g$ and $\dim_k H^1(\Omega^1_{X_K}) = 1$.

There are authors who call $\dim_k H^0(\Omega^1_{X_K})$ the analytic genus of X_K, so then one would say analytic genus = arithmetic genus.

Exercise 3.37. (1) Show that $\deg(\Omega^1) = 2g - 2$.

(2) If D is a divisor of degree $> 2g - 2$, can you describe $h^0(\mathcal{O}(D))$ and $h^1(\mathcal{O}(D))$?

If you are into algebraic or differential topology, then you can have a try at the following exercise.

Exercise 3.38. Look up "Euler characteristic" of "closed oriented surfaces" and "Poincaré–Hopf theorem", and show that the "topological genus" of a curve over $\mathbb{C}$ is given by g as well.

So here we are, topological genus = arithmetic genus = analytic genus! Let's stop here and have a beer (or coffee, or tea, or coke, but never, never pepsi).

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