MIDTERM SOLUTION

1. True-False.

(1) This is false. For example, take $\vec{u} = \vec{i}$ and $\vec{v} = \vec{j}$. Then $|\vec{u} - \vec{v}| = \sqrt{2}$ and $|\vec{u}| - |\vec{v}| = 0$.

(2) This is true. In fact, for any vector $\vec{u}$ and any nonzero vector $\vec{w}$, one can find a vector $\vec{v}$ so that $\vec{v} \times \vec{w} = \vec{u}$ *if and only if* $\vec{u}$ is *perpendicular to* $\vec{w}$.

In this question, we may take $\vec{v} = \langle 0, 3, 2\rangle$. One checks easily that $\langle 0, 3, 2\rangle \times \langle 1, 1, 1\rangle = \langle 1, 2, -3\rangle$

(3) This is false. We divide the equation by -1 to bring it into the forms discussed in class: $-x^2 + y^2 + z^2 = 1$. Now the right-hand is 1, the left-hand has three square terms, hence it is the non-degenerate case. Since there are two positive signs and one negative sign, we see that this is a hyperboloid of one sheet.

(4) This is false. Apply product rule twice, we should get

$$\frac{d}{dt}(\vec{a} \times (\vec{b} \times \vec{c})) = \vec{a}' \times (\vec{b} \times \vec{c}) + \vec{a} \times (\vec{b}' \times \vec{c}) + \vec{a} \times (\vec{b} \times \vec{c}').$$

For counterexample, just take $\vec{a}(t) = t\,\vec{i}$, $\vec{b}(t) = t\,\vec{i}$ and $\vec{c}(t) = t\,\vec{j}$.

(5) This is false. As we said in the class, the curvature of a circle is the reciprocal of its radius. Hence the curvature is $1/2$.

2. In the following, let $\vec{v} = \langle 1, 1, 1\rangle$ and $\vec{w} = \langle -3, 1, -5\rangle$.

(a) $\langle -6, 2, 4\rangle$ (+5 points).

(b) $\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ (+3 points) $= \frac{-7}{35}\langle -3, 1, -5\rangle = \langle \frac{3}{5}, \frac{-1}{5}, 1\rangle$ (+2 points).

(c) cos(acute angle) $= \frac{|\vec{v}\cdot\vec{w}|}{|\vec{v}||\vec{w}|}$ (+3 points) $= \frac{7}{\sqrt{3}\sqrt{35}} = \frac{\sqrt{105}}{15}$ (+2 points).
The opposite answer only earns you 2 points.

(d) First we find a normal vector perpendicular to such a plane:

$$\vec{n} = \vec{v} \times \vec{w} \text{ (+3 points) } = \langle -6, 2, 4\rangle$$

Hence an equation is given by

$$-6x + 2y + 4z = d \text{ (+1 point)}.$$

In order for the plane to pass through the point $(1, 2, 3)$, we see that $d = 10$. Hence an equation is $-6x + 2y + 4z = 10$ (+1 point).

(e) distance $= \frac{|d|}{|\vec{n}|}$ (+3 points) $= \frac{10}{\sqrt{56}} = \frac{5}{\sqrt{14}} = \frac{5\sqrt{14}}{14}$ (+2 points).

3. Let C be the curve on the xy-plane defined by $x^2 - 2y^2 = 1$. Consider rotating C around the x-axis, resulting a rotation surface in the 3-dimensional space. Denote this surface by S.

(a) Use the fact that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ (+ 5 points).
The equation becomes $r^2(\cos^2(\theta) - 2\sin^2(\theta)) = 1$ (+3 points).
Therefore we get $r = \frac{1}{\sqrt{\cos^2(\theta)-2\sin^2(\theta)}}$ (+2 points).

(b) Suppose the point $P(x_0, y_0, z_0)$ is on the rotating orbit of the point $Q(x_1, y_1)$ and that Q is on the curve C. Then we have three equations:

(1) $x_1^2 - 2y_1^2 = 1$, because Q is on the curve C (+2 points);
(2) $x_0 = x_1$, since Q rotates around x-axis and passes P (+2 points) and;
(3) $|y_1| = \sqrt{y_0^2 + z_0^2}$, since P and Q have the same distance to x-axis (+2 points).

Plug (2) and (3) into the equation in (1) gets us $x_0^2 - 2y_0^2 - 2z_0^2 = 1$ (+4 points).

(c) This is a hyperboloid of two sheets (+5 points).

(d) It is a fact that there is no line lying on a hyperboloid of two sheets.
In our situation, suppose there is a line lying on $x^2 - y^2 - z^2 = 1$ passing through $(1, 0, 0)$, given by $x = 1 + at, y = bt, z = ct$ and not all of a, b, c are 0 (+2 points).

Plug into the equation gives $(1+at)^2 - (bt)^2 - (ct)^2 - 1 = 2at + (a^2 - b^2 - c^2)t^2 = 0$ (+2 points). In order for this to hold for all t, we have $2a = 0$ and $a^2 - b^2 - c^2 = 0$ (+2 points). The first equation implies $a = 0$, hence the second equation becomes $b^2 + c^2 = 0$ which forces both of b and c to be 0 (+2 points). This contradicts to the assumption that not all of a, b and c are 0. Hence there is no line satisfying all the conditions asked for in the question (+2 points).

If you say there is no such a line without a justification, you will get 2 points.

4. $\vec{r}(t) = \sin(t)\,\vec{i} + t\,\vec{j} + \cos(t)\,\vec{k}$.

(a) $\vec{r}'(t) = \langle \cos(t), 1, -\sin(t) \rangle$ (+1 point).
$|\vec{r}'(t)| = \sqrt{\cos^2 + 1 + \sin^2} = \sqrt{2}$ (+1 point).
Hence $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ (+2 points) $= \frac{1}{\sqrt{2}}\langle \cos(t), 1, -\sin(t) \rangle$ (+1 point).

(b) $\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), 0, -\cos(t) \rangle$ (+1 point)
$|\vec{T}'(t)| = \frac{1}{\sqrt{2}}$ (+1 point).
Hence $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$ (+2 points) $= \frac{1}{2}$ (+1 point)
You can use other ways to compute the curvature, the formula worth 2 points, the intermediate steps worth 2 points and the final answer worth 1 point.

(c) $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ (+2 points) $= \langle -\sin(t), 0, -\cos(t) \rangle$ (+1 point).
$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ (+1 point) $= \frac{1}{\sqrt{2}}\langle -\cos(t), 1, \sin(t) \rangle$ (+1 point).

(d) The point $(0, \pi, -1)$ corresponds to $t = \pi$ (+1 point).
$s = f(t) = \int_\pi^t |\vec{r}'(x)| dx = \sqrt{2}(t - \pi)$ (+1 point).
The inverse function is given by $t = f^{-1}(s) = \frac{s}{\sqrt{2}} + \pi$ (+1 point).
Hence the reparametrization is given by $\vec{u}(s) = \vec{r}(f^{-1}(s))$ (+1 point) $= \langle \sin(\frac{s}{\sqrt{2}} + \pi), \frac{s}{\sqrt{2}} + \pi, \cos(\frac{s}{\sqrt{2}} + \pi)$ (+1 point).

5. A particle is moving in the space with position vector function $\vec{r}(t)$ with initial speed $v(0) = 1$. Suppose we know its acceleration is given by $\vec{a}(t) = e^t\vec{T}(t) + e^t\vec{N}(t)$. Compute the normal component of the third derivative of $\vec{r}(t)$ (i.e., compute $\vec{r}(t)''' \cdot \vec{N}(t)$). (10 points)

$v'(t) = a_T = e^t$ (+1 point), hence $v(t) = v(0) + \int_0^t e^x dx = e^t$ (+1 point).

$\kappa(t)v^2(t) = a_N = e^t$ (+1 point), hence $\kappa(t) = e^{-t}$ (+1 point).

Furthermore we have $\vec{T} \cdot \vec{N} = 0$ (+1 point), $\vec{N} \cdot \vec{N} = 1$ (+1 point), $\vec{N}' \cdot \vec{N} = 0$ (+1 point) and $\vec{T}'(t) = v(t)\kappa(t)\vec{N}(t) = \vec{N}(t)$ (+1 point).

Hence $\vec{r}'''(t) \cdot \vec{N}(t) = (e^t\vec{T}(t) + e^t\vec{N}(t))' \cdot \vec{N}(t) = e^t\vec{T}'(t) \cdot \vec{N}(t) + e^t\vec{N}(t) \cdot \vec{N}(t) = 1 + e^t$ (+2 points).

Bonus question: write down a curve parametrized by arc length with curvature function given by $\kappa(s) = s$. (10 points, no partial credit)

Let's make a plane curve parametrized by arc length with this prescribed curvature. Suppose $\overrightarrow{T}(s) = \langle \cos(\theta(s)), \sin(\theta(s)) \rangle$, then $\frac{d\overrightarrow{T}}{ds} = \theta'(s)\langle -\sin(\theta(s)), \cos(\theta(s))\rangle$. Hence we have $\theta'(s) = s$, and we may set $\theta(s) = \frac{1}{2}s^2$. Finally, we have $\overrightarrow{r}(s) = \int_0^s \overrightarrow{T}(x)dx = \langle \int_0^s \cos(\frac{1}{2}x^2)dx, \int_0^s \sin(\frac{1}{2}x^2)dx \rangle$.