THE FIRST "METRIC" COHOMOLOGY GROUPS OF SMOOTH AFFINOID SPACES

DIE ERSTE "METRISCHE" KOHOMOLOGIEGRUPPE GLATTER AFFINOIDER RÄUME

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Introduction and Results

In the paper [5] of the author a convergence theorem for 0-cochain of affinoid functions was proved, which among others gives a proof other than the one of Lütkebohmert for the theorem of "Thullen-Remmert-Stein" about the continuability of a holomorphic set in an exception set of the same dimension. L. Gerritzen told the author that the aforementioned convergence theorem is a corollary of a yet to be proved vanishing theorem for the cohomology $H^1(X, \check{\mathcal{O}})$ which can be understood. Denote $H^1_\rho(\mathcal{U})$ the first cohomology group of the complex $C^\bullet_\rho(\mathcal{U})$ which associates with an (finite, affinoid) open cover of an affinoid space X the functions of norm $< \rho$ on respective intersections. In this work we show the following:

Theorem 1. *For any $\rho \in \mathbb{R}^*_+$ and any open cover $\mathcal{U}$ of unit polycylinder E^n, the group $H^1_\rho(\mathcal{U})$ vanishes. In particular the same applies to $H^1_\rho(E^n)$*[1].

Theorem 2. *Let X be a smooth (absolutely regular) affinoid space over k. Then there is a constant $c \in k$ with $0 < |c| \leq 1$ so that*

$$c \cdot H^1_\rho(X) = 0$$

for all ρ.

The example $V(Y^2 + X^3 + aX)$ where $|a| < 1$ of "deformed" cubic cuspidal shows (S. Bosch) that one cannot choose $c = 1$ in general and suggests that this phenomenon has something to do with the singularity on the affine model[2]. In fact we have[3],

Theorem 2'. *Let X be an affinoid space over an algebraically closed field k with smooth affine model $\widetilde{X}$. Then*

$$H^1_\rho(X) = H^1_1(X) = 0.$$

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[1]Translator's note: perhaps the author meant the following. Let us consider the sheaf of functions of norm $< \rho$ on E^n, then by general nonsense in topos theory we know that H^1 of this sheaf can be computed as colimit of Čech H^1 of open covers. Therefore by the statement before, we see that H^1 of this sheaf vanishes also. In fact, according to (reference de Jong, Van der Put) the sheaf cohomology on a rigid space can always be computed by Čech cohomology

[2]Translator's note: maybe the author was saying the non-vanishing of $H^1_\rho(X)$ is due to the affine model being singular.

[3]Author's note: in a lecture Sir Gerritzen conjectured that Theorem 2 is equivalent to Theorem 2'.

With a Riemannian removability argument, i.e., ball theorem (Kugelsatz) the prerequisite "smooth" could be slightly weakened to "absolutely normal". On the other hand it seems hard to prove the analogue of Theorem 2 for higher cohomology groups. So far the author has no concrete idea of how to remove the condition of "absolute". A vanishing theorem for H^2 would be interesting, e.g. if one considers the relation between Picard groups of X and $\widetilde{X}$ under less special conditions as studied in [6], [7].

§ 1. Definitions and Preparations

1. Let $X = (X, \mathcal{A}, \cdot)$ be a rigid space as in [4], so there is an admissible open cover of X by affinoid subspace, in other words an atlas of X ([8]). We use A to denote the ring of analytic functions on X, and $\mathring{A} \subset A$ the subring of functions with norm ≤ 1. Let $\mathcal{U}$ be an open cover of X, denote $C^\bullet(\mathcal{U})$ the A-complex associated with $(\mathcal{A}, \cdot)$. For $\rho \in \mathbb{R}_+^*$, let[4]

$$C_\rho^q(\mathcal{U}) := \left\{ (f_i)_{(i)} \,\middle|\, (f_i)_{(i)} \in C^q(\mathcal{U}), |(f_i)_{(i)}| < \rho \right\},$$

where $|\quad|$ denotes the maximum of spectral norm. $C_\rho^\bullet(\mathcal{U})$ is a $\mathring{A}$-complex. In the case of $\rho = 1$ we denote it by $C^\bullet(\mathcal{U}, \check{A})$. We use the usual notation $Z_\rho^\bullet(\mathcal{U})$ (resp. $B_\rho^\bullet(\mathcal{U})$) to denote the subcomplex of cocycles (resp. coboundaries), and denote their quotient by $H_\rho^\bullet(\mathcal{U})$. Finally we set

$$H_\rho^\bullet(X) := \varinjlim_{\mathcal{U}} H_\rho^\bullet(\mathcal{U})$$

where the limit is taking over the system of all open covers.

2. From now on X will be an affinoid. We denote $\mathcal{U} \vee \mathcal{B}$[5] the canonical common refinement of two open covers and write $\mathcal{U} \gg \mathcal{B}$ if $\mathcal{U}$ is finer than $\mathcal{B}$. We use $\mathcal{B} \cap U$ to denote the induced open cover of a subspace $U \subset X$.

Suppose for $\nu = 1, \dots, n$ we have

$$f_\nu \in A, \epsilon_\nu \leq \epsilon'_\nu \in \sqrt{|k^*|},$$

then we set

$$\mathcal{U}(f, \underline{\epsilon}, \underline{\epsilon'}) := \bigvee_{\nu=1}^{n} (\{|f_\nu| \leq \epsilon'_\nu\}, \{|f_\nu| \geq \epsilon_\nu\}).$$

If $\underline{\epsilon} = \underline{\epsilon'}$ these form the Laurent open cover ([11]). Given $f_1, \dots, f_n \in A$ without common zero and $1 \leq \rho \in \sqrt{|k^*|}$, then call

$$\mathcal{B}_\rho := (V_{\nu,\rho})_{(\nu=1,\dots,n)}, V_{\nu,\rho} := \{|f_1|, \dots, |f_n| \leq \rho |f_\nu|\}$$

[4]Translator's note: the author used $C^q(\mathcal{U})$ but the translator has changed the notation to indicate the relevance of ρ.

[5]Translator's note: due to the translator's mistake, the curly V in the original paper (standing for an open cover) has been wrongly denoted as curly B here. The reader should be aware that from now on there will be plenty occasions where the elements of curly B are called V_i.

the *standard rational cover of* X *of* $\underline{f}$[6] *with radius* ρ. More generally given $\mathcal{B} = (V_i)_{(i)}$ any rational open cover of X, i.e. an open cover with rational subdomains

$$V_i = \left\{ \left| f_1^{(i)} \right|, \ldots, |f_{m_i}^{(i)}| \leq |g_i| \right\},$$

then for $1 \leq \rho \in \sqrt{|k^*|}$ we set

$$V_{i,\rho} := \left\{ \left| f_1^{(i)} \right|, \ldots, |f_{m_i}^{(i)}| \leq \rho |g_i| \right\}.$$

When $\rho > 1$ we call $\mathcal{B}_\rho := (V_{i,\rho})_{(i)}$ an *inflated open cover* of X. The inflated open covers will be used for two essential reductions.

We treat the first one immediately. We recall the following

Satz 1. [7][11],[12][8] *Let* $\mathcal{B}$ *be an open cover of* X *with special subdomains*

$$V_{i,\underline{n}} = \left\{ \left| f_j^{(i)} \right|^{n_j} \leq 1, j = 1, \ldots, m_i \right\}, \underline{n} \in \{-1, 1\}^{m_i}$$

and denote the family $f_j^{(i)}$ *by* $\underline{f}$. *Then the Laurent cover* $\mathcal{U} = \mathcal{U}(\underline{f}, \underline{1}, \underline{1})$ *has the property that for any* $U \in \mathcal{U}$ *the open cover* $\mathcal{B} \cap U$ *is formal.*

From this Satz one obtains

Satz 2. *Let* $\mathcal{B}_\rho = (V_{i,\underline{n},\rho})_{(i)}$ *be an inflated open cover of* X. *Then there is an open cover* $\mathcal{U}$ *of* X *with special subdomains, such that for any* $U \in \mathcal{U}$ *there exists a formal open cover* $\mathcal{M}$ *of* U *so that*

$$\mathcal{B} \cap U \gg \mathcal{M} \gg \mathcal{B}_\rho \cap U$$

Proof. Let

$$V_i = \left\{ \left| f_1^{(i)} \right|, \ldots, |f_{m_i}^{(i)}| \leq |g_i| \right\}.$$

We may choose $m \in \mathbb{N}$ so big such that

$$V_{i,\rho} \cap \left\{ |g_i| \leq \rho^{-m} \right\} = \varnothing, |g_i| \leq \rho^m$$

for all i and set

$$P_{i,-m} := \left\{ |g_i| \leq \rho^{-m} \right\}, P_{i,\mu} := \left\{ \rho^{\mu-1} \leq |g_i| \leq \rho^\mu \right\} \text{for} - m < \mu \leq m,$$

$$\mathcal{P}_i = (P_{i,\mu})_{(|\mu| \leq m)}, \quad \mathcal{P} := \bigvee_i \mathcal{P}_i.$$

Let $P \in \mathcal{P}$ indexed as $(\mu_i)_{(i)}$, then we set

$$W_{i,P} := \begin{cases} P \cap \left\{ \left| f_1^{(i)} \right|, \ldots, \left| f_{m_i}^{(i)} \right| \leq \rho^{\mu_i} \right\} & \text{for } \mu_i > -m; \\ \varnothing & \text{otherwise.} \end{cases}$$

[6] Translator's note: the author used f without underline, here we add an underline to indicate it is associated with an n-tuple of functions (without common zero). There are also other typos in this article, we will just make the reasonable correction without note. The reader is encouraged to read with a grain of salt.

[7] Translator's note: perhaps we should use the word Proposition instead of Satz as we are translating. But the translator decided not to do that for some reason.

[8] Translator's notice: the translator was not able to find any discussion of formal open cover in [12]. In [11, Page 259] formal open cover of an affinoid is introduced.

For all we have

$$V_i \cap P \subset W_{i,P} \subset V_{i,\rho} \cap P,$$

in particular $\mathcal{M}_P := (W_{i,P})_{(i)}$ is an open cover of P with special subdomains. According to Satz 1, there is a Laurent cover $\mathcal{L}_P$ of X such that for any $L \in \mathcal{L}_P \cap P$ the open cover $\mathcal{M}_P \cap L$ is formal. Finally one can set

$$\mathcal{U} := \mathcal{P} \vee (\bigvee_{P \in \mathcal{P}} \mathcal{L}_P)$$

and for $U = P \cap (\ldots) \in \mathcal{U}$ we set $\mathcal{M} := \mathcal{M}_P \cap U$. □

3. We formulate another important property of inflated covers in

Satz 3. *Let $\mathcal{U}_\epsilon$ be an inflated open cover of an affinoid space Y and $\Phi : Y \to X$ a finite morphism. Then there is an open cover $\mathcal{M}$ of X with $\Phi^{-1}(\mathcal{M})' \gg \mathcal{U}_\epsilon$.*

We denote by $\mathcal{B}'$ the refinement of $\mathcal{B}$ by decomposing elements of $\mathcal{B}$ into its connected components.

Proof. (c.f. [5, Satz 4.4]) We do induction in n, where $\mathcal{U} = (U_\nu)_{(\nu=1,\ldots,n)}$ with trivial beginning $n = 1$. Assuming it for $n-1$ with $n \geq 2$, let

$$U_n = \{|f_1|, \ldots, |f_{m-1}| \leq |f_m|\}.$$

Let $\rho, \sigma \in \sqrt{|k^*|}$ with $\sigma > \rho > 1$ and $\sigma \cdot \rho \leq \epsilon$. Then there is a $\mathcal{B}$ with

$$\begin{aligned} V_\mu &:= \{|f_1|, \ldots, |f_{m-1}|, \sigma \cdot |f_m| \leq \rho |f_\mu|\}, \quad \mu = 1, \ldots, m-1 \\ V_m &:= \{|f_1|, \ldots, |f_{m-1}| \leq \rho \cdot \sigma \cdot |f_m|\} \end{aligned}$$

a rational standard open cover with radius > 1. Now by [5, Satz 4.2] there is an open cover $\mathcal{M} = (W_i)_{(i)}$ of X with $\Phi^{-1}(\mathcal{M})' \gg \mathcal{B}$. We have

$$V_m \subset U_{n,\epsilon}, \quad V_\mu \cap U_n = \varnothing, \quad \mu = 1, \ldots, m-1.$$

Let $\Phi^{-1}(W_i)_j$ be the connected component of $\Phi^{-1}(W_i)$, so we can choose a refining map $\mu(\cdot,\cdot)$ with $\Phi^{-1}(W_i)_j \subset V_{\mu(i,j)}$. For $\mu(i,j) < m$ the induction hypotheses can be applied to

$$\mathcal{U} \cap \Phi^{-1}(W_i)_j, \quad \Phi^{-1}(W_i)_j \to W_i.$$

Let $\mathcal{M}_{i,j}$ be a corresponding open cover of W_i, then we set

$$\mathcal{M}_i := \bigvee_{\mu(i,j)<m} \mathcal{M}_{i,j}.$$

The open cover of X formed by all elements of all $\mathcal{M}_i$ is what we are looking for.[9] □

[9]Translator's note: we need to distinguish two cases depending on whether there is an index i such that for all j we have $\mu(i,j) = m$. If no such an i exists, then we are done. Otherwise we just add in W_i.

4. Finally we note a result of Kiehl ([10, proof of Satzes 1.12 and conclusion]), which is inevitable for the proof of Theorem 2.

Satz 4. *Let X be a smooth affinoid space. Then there is an atlas of X whose members are irreducible subdomains (Y,B) with the following property: for any $y \in Y$ there exists a finite morphism:*

$$\phi : Y \to E^n = (\operatorname{Max} T, T), \quad n = \dim Y,$$

and an element $b \in \mathring{B}$, whose minimal polynomial $\Omega \in \mathring{T}[Z]$ has discriminant f with $f(\phi(y)) \neq 0$ and so that

$$T_f[Z]/(\Omega) \xrightarrow{\sim} B_{\phi^*(f)}.$$

Remark 1. Specifically for $\epsilon \in \sqrt{|k^*|}$

$$Y_{|\phi^*(f)|\geq\epsilon} \to V(\Omega) \cap E^{n+1}_{|f|\geq\epsilon}.$$

The analogue for affine spaces over an arbitrary ground field κ (i.e. space $\operatorname{Spec} A$, where A is a κ-algebra of finite type) might also be true. We restrict ourselves to the case which is required later by the proof of Theorem , where κ is algebraically closed.

Satz 4'. *Let $Z = (Z, B)$ be an irreducible smooth affine space over κ. Then for every point $z_0 \in Z$ there exists a finite morphism*

$$\phi : Z \to \mathbb{A}^d, \quad d = \dim Z,$$

so that the fiber $\phi^{-1}(\phi(z_0))$ is reduced. Specifically there is an element $b \in B$, whose minimal polynomial $\Omega \in \kappa[X][W]$ with respect to

$$\phi^* = \kappa[X_1, \dots, X_d] \to B$$

has discriminant f with $f(\phi(z_0)) \neq 0$ and

$$\kappa[X]_f[W]/(\Omega) \xrightarrow{\sim} B_{\phi^*(f)}.$$

Hilfssatz 5. *Let (Z,B) be a given reduced closed subspace of dimension $\leq d$ in $\mathbb{A}^{d+n}_\kappa = \operatorname{Spec}(\kappa[X_1,\dots,X_{d+n}])$ containing the “origin” z_0, suppose $n \geq 1$ and $d_1,\dots,d_{n+d-1} \in \mathbb{N}^+$. For any $a \in \kappa^{d+n-1}$ we consider projection (depending on a)*

$$\pi : Z \to \mathbb{A}^{d+n-1},$$

given by

$$\pi^*(X_\nu) := (X_\nu - a_\nu \cdot X_{d+n}^{d_\nu})|_Z, \quad \nu = 1, \dots, d+n-1.$$

If π is finite, denote $Y = (Y, A)$ the image of Z (with its reduced structure) and $\phi : Z \to Y$ the morphism induced by π. Then we have

- (α) *There is a (Zariski-) open subset $\varnothing \neq U \subset \mathbb{A}^{d+n-1}$ so that for any $a \in U$ the morphism π is finite.*
- (β) *If $\dim Z < d$ or $n > 1$, then there is an open subset $\varnothing \neq U \subset \mathbb{A}^{d+n-1}$ such that for any $a \in U$ we have $\phi^{-1}(\phi(z_0)) = \{z_0\}$.*
- (γ) *If all $d_\nu > 1$, π is finite, $\phi^{-1}(\phi(z_0)) = \{z_0\}$ and the images of $X_1,\dots,X_d$ form a regular system of parameters of B_{z_0}, then the images of $X_1,\dots,X_d$ in $A_{\phi(z_0)}$ also form a regular system of parameters. In particular, Y is also regular at $\phi(z_0)$ and $\widehat{A}_{\phi(z_0)} \approx \widehat{B}_{z_0}$.*

(δ) *If $n = d_1 = \ldots = d_{d+n-1} = 1$, Z is irreducible of dimension d and regular at z_0. Suppose $f \in \kappa[X_1, \ldots, X_{d+1}]$ is a prime element precisely defining Z such that*

$$\frac{\partial f}{\partial X_\nu}(z_0) = \delta_{\nu,d+1}, \quad 1, \ldots, d+1,$$

then there is an open subset $\varnothing \neq U \subset \mathbb{A}^d$ such that for such $a \in U$ the fiber $\pi^{-1}(\pi(z_0))$ is reduced and Z is regular at all of its points[10].

Both of Satz 4' and Hilfssatz 5 are classical algebraic geometry/commutative algebra, let us not translate the proof here.

§ 2. Proof of Theorems

1. In this section, let $X = (X, A)$ be an affinoid space. In individual subsections there will be more conditions on it.

From Tate's proof of acyclicity for formal open covers ([12, Lemma 8.4]) we get

Satz 5. *For each formal open cover $\mathcal{U}$ of X and $\rho \in \mathbb{R}_+^*$, the supplement complex $C_\rho^\bullet(\mathcal{U})$ given by $A_\rho \coloneqq \{a \mid |a| < \rho\}$ is acyclic.*

Proof. Following the notations in [12, page 274–275], here we also get a resolution

$$0 \to C_\rho^\bullet(\mathcal{P}) \xrightarrow{\alpha} C_\rho^\bullet(\mathcal{P}) \to C_\rho^\bullet(\mathcal{U}) \to 0,$$

because $1 - f \cdot S$ is also non-zero divisor in $\widetilde{A}[S]$, hence the homotopy to $1 - f \cdot S$ of $A_\rho\langle S\rangle$ is isometric with respect to $||$, and secondly after [2, Satz 5.1]

$$A_\rho\langle S_\sigma\rangle \to (A\langle f_\sigma^{-1}\rangle)_\rho$$

is surjective. Finally the null-homotopy given by Tate

$$s : C^\bullet(\mathcal{P}) \to C^{\bullet-1}(\mathcal{P})$$

not from $C_\delta^\bullet(\mathcal{P})$, how to get directly from the defining formula sees, if one still observes, that

$$\left|\sum a^{(\underline{\nu})} S^{\underline{\nu}}\right| = \max \left|a^{(\underline{\nu})}\right|.$$

□

The following argument allows us again to descend an inflated cover to "any" covers.

Satz 6. *Let $\rho \in \mathbb{R}_{>0}^*$, $q \in \mathbb{N}^+$ be arbitrary. If there is a $c \in k$ with $0 < |c| \leq 1$, so that for any inflated open cover $\mathcal{U}_\epsilon$ of X and any $f \in Z_\rho^q(\mathcal{U}_\epsilon)$ there exists an open cover $\mathcal{B} \gg \mathcal{U}_\epsilon$, such that*

$$c \cdot f|_{\mathcal{B}} \in B_\rho^q(\mathcal{B}),$$

then

$$c \cdot H_\rho^q(X) = 0.$$

[10] Translator's note: perhaps the author meant regular at all of $\pi^{-1}(\pi(z_0))$.

Proof. The system of all rational covers is cofinal due to the main result of [8]. Let $\mathcal{U}$ be an arbitrary open cover of X. For $1 < \epsilon \in \sqrt{|k^*|}$ consider the diagram

$$\begin{array}{ccc} C^{q-1}(\mathcal{U}_\epsilon) & \xrightarrow{\partial_\epsilon^{q-1}} & Z^q(\mathcal{U}_\epsilon) \\ \downarrow \beta^{q-1} & & \downarrow \beta^q \\ C^{q-1}(\mathcal{U}) & \xrightarrow{\partial^{q-1}} & Z^q(\mathcal{U}) \end{array}$$

where the vertical arrows are restrictions and all A-modules are equipped with the canonical Banach topology. Since each U_i is a Weierstrass domain in $U_{i,\epsilon}$, β^{q-1} has dense image; as ∂_ϵ^{q-1} and ∂^{q-1} are open, therefore β^q also has dense image. Now for any $f \in Z_\rho^q(\mathcal{U})$, there is a $g \in C_\rho^{q-1}(\mathcal{U})$ and $h \in Z^q(\mathcal{U}_\epsilon)$ with

$$f - \partial^{q-1}(g) = \beta^q(h).$$

By decreasing $\epsilon > 1$ we can even assume that $h \in Z_\rho^q(\mathcal{U}_\epsilon)$, see [5, Hilssatz 4.3][11]. Therefore it suffices to show that $c \cdot h$ becomes zero after restricting to a refinement of $\mathcal{U}_\epsilon$. This is the case by our condition. □

2. We now show our first “substantial” result.

Satz 7. *Let $X = E^n$ be the unit polycylinder, then for each Laurent cover $\mathcal{U} = \mathcal{U}(\underline{f}, \underline{1}, \underline{1})$ of X we have a short exact sequence*

$$0 \to A \xrightarrow{\iota} C^0(\mathcal{U}) \xrightarrow{\partial^0} B^1(\mathcal{U}) = Z^1(\mathcal{U}) \to 0.$$

The homomorphism ι has a left inverse π such that the bijection $\partial^0|_{\mathrm{Ker}\ \pi}$ is an isometry with respect to $|\quad|$.

We first show that Theorem 1 follows from this Satz. According to Satz 1 and Satz 7, we see that $H_\rho^1(\mathcal{U}) = 0$[12] for any open cover $\mathcal{U}$ of E^n with special subdomains. Now let $\mathcal{B}$ be some open cover by rational subdomains[13] and $f \in Z_\rho^1(\mathcal{B})$. As in the proof of Satz 6 above, one can reduce the proof of the claim $f \in B_\rho^1(\mathcal{B})$ to the case where f is the restriction of an element $f' \in Z_\rho^1(\mathcal{B}_\epsilon)$ with some $\epsilon > 1$. Let $\mathcal{U}$ be as in Satz 2 and to $U \in \mathcal{U}$ one get a formal open cover $\mathcal{M}$ of U with the property in the mentioned Satz. Now Satz 5 gives $f'|_{\mathcal{M}} \in B_\rho^1(\mathcal{M})$, hence also

$$f|_{\mathcal{B} \cap U} \in B_\rho^1(\mathcal{B} \cap U).$$

Because $H_\rho^1(\mathcal{U}) = 0$, it follows — c.f. the proof of Hilfssatzes 3 — that $f \in B_\rho^1(\mathcal{B})$[14]. An almost identical argument settles the case of an arbitrary open cover[15] $\mathcal{B} = (V_i)_{(i=1,\dots,s)}$.

[11]Translator’s note: unfortunately it seems that the reference is not available online. However this statement seems easy to prove anyway. It is crucial that we are considering cochains with norm less than (no “or equal to”) a fixed real number ρ.

[12]Translator’s note: here we need the auxiliary Lemma A.1.

[13]Translator’s note: this is not necessarily a rational cover, at least not from the definition.

[14]Translator’s note: once again, we need to use the auxiliary Lemma A.1

[15]Translator’s note: the author should say it is an open cover by affinoid subdomains in order to apply the Theorem of Gerritzen and Grauert cited.

Due to [8, Satz 3.6], each V_i is a union of rational subdomains $V_i^{(j)},\ j = 1,\dots,m_i$ of E^n. According to previously proved case[16], to each $(f_{il})_{(i,l)} \in Z^1_\rho(\mathcal{B})$ there exist

$$(f_i^{(j)})_{(i,j)} \in C^0_\rho((V_i^{(j)})_{\binom{i=1,\dots,r}{j=1,\dots,m_i}})$$

with

$$f_{il} = f_i^{(j)} - f_l^{(j')} \quad \text{over} \quad V_i^{(j)} \cap V_l^{(j')},$$

so that the $f_i^{(j)},\ j = 1,\dots,m_i$ glue to an affinoid function f_i on V_i and so that $(f_i)_{(i)} \in C^0_\rho(\mathcal{B})$ whose coboundary is $(f_{i,j})_{(i,j)}$.

Proof of Satz 7. Let us denote $n = d+1 > 0$, i.e. $A = T_d\langle Y\rangle$. We can assume that $\mathcal{U}$ is of the form $\mathcal{U}(\underline{\omega},\underline{\epsilon},\underline{\epsilon})$, where $1 \geq \epsilon_\mu \in \sqrt{|k^*|}$ and $\omega_\mu \in \mathring{T}_d[Y],\ \mu = 1,\dots,m$ are Weierstrass polynomials of degree $s_\mu > 0$[17]. The projector π is obtained in the following way. Let

$$\mathcal{U}_{2^m} := \{|\omega_\mu| \geq \epsilon_\mu,\ \mu = 1,\dots,m\}$$

and $(f_i)_{(i)} \in C^0(\mathcal{U})$. Then by [5, Satz 2.3] we have a unique decomposition[18]

$$f_{2^m}|_{\{|\omega|=1\}} = h + \sum_{\nu<0} r_\nu \omega^\nu,$$

where $\omega := \prod \omega_\mu$. Finally we define

$$\pi((f_i)_{(i)}) := h.$$

By construction π is a contracting projector. We now prove that[19]

$$\left|(f_i)_{(i)}\right| \leq \left|\partial(f_i)_{(i)}\right| \quad \text{if} \quad h = 0.$$

By passing to the residue class of T_d it suffices to prove the inequality when $d = 0$, after a base change of the ground field we can also assume that k is algebraically closed. Let $\mathbb{P} = \mathbb{P}^1(k)$ with its usual holomorphic structure, and let

$$U' := \{\epsilon_\mu \leq |\omega_\mu| \leq \infty,\ \mu = 1,\dots,m\}$$

an affinoid subdomain of $\mathbb{P}$ with the Atlas $\mathcal{U}(\omega^{-1},1,1)$. We have

$$U'_{|\omega^{-1}|\geq 1} \subset U_{2^m} \subset U'.$$

Thus f_{2^m} naturally has an affinoid continuation f' on U'[20] such that $f'(\infty) = 0$. Then $(f_i)_{(1\leq i<2^m)}$ and f' defines a 0-cochain associated with the open cover $(U_i)_{(1\leq i<2^m)} \cup U'$ of $\mathbb{P}$. Because

$$U_i \cap U' = U_i \cap U_{2^m} \quad \text{for} \quad i < 2^m$$

it suffices to show:

[16]Translator's note: the previously proved case exactly is open cover by rational subdomains.

[17]Translator's note: the author forgot to mention that the ω_μ's should have norm 1.

[18]Translator's note: since the reference is not available online, the translator proved the related statement in the appendix, see Lemma A.2.

[19]Translator's note: the opposite inequality is obvious.

[20]Translator's note: the translator does not understand the author's expression. Fortunately, the translator has a simple argument to show that f_{2^m} can be continued to U', see Lemma A.3.

Satz 8. *Let k be algebraically closed and $(U_i)_{(1\le i\le n)}$ be an open cover of $\mathbb{P}$. Then for any j_0 and $x_0\in U_{j_0}$, then projector p with*

$$p((f_i)_{(i)}) := f_{j_0}(x_0)$$

splits the following exact sequence

$$0 \longrightarrow k \underset{\overset{p}{\dashleftarrow}}{\longrightarrow} C^0(\mathcal{U}) \xrightarrow{\partial^0} B^1(\mathcal{U}) \longrightarrow 0,$$

and $\partial|_{\mathrm{Ker}\ p}$ is an isometry.

Proof. We may assume that all U_i are connected and $n>1$. Without loss of generality let $j_0=n,\ x_0=\infty$. After renumbering there is an m with $1\le m<n$ such that

$$\infty\in U_i \iff m<i\le n.$$

Let $\rho\in|k^*|$ be big enough, so that

$$\bigcup_{i=1}^{m} U_i \subset \{|Y|\le\rho\} =: K, \quad \mathbb{P}\backslash\check{K}\subset\bigcap_{i=m+1}^{m} U_i.$$

Following the notations in [5, § 2.3] for $1\le i\le m$ set

$$U_i = D^{(i)}\backslash\bigcup_{l=1}^{m_i}\check{D}^{(i,l)},\ m_i\ge 0. \tag{*}$$

There is a similar presentation for $m<i\le n$, where $\check{D}^{(i,l)}\subset k$[21] are disjoint “open” discs and $D^{(i)}=\mathbb{P}$, especially $m_i>0$. For each i let

$$f_i = h_i + f_{i,-}$$

be the decomposition corresponding to (*), where h_i is holomorphic on $D^{(i)}$ and $f_{i,-}$ is holomorphic on

$$\mathbb{P}\backslash\bigcup_{l=1}^{m_i}\check{D}^{(i,l)} \quad\text{with}\quad f_{i,-}(\infty)=0.$$

If we set

$$\mathcal{B} := \mathcal{U}\cap K,\ (g_i)_{(i)} := (f_i)_{(i)}|_{\mathcal{B}},$$

then by [5, Satz 2.5][22]

$$g_i(K,V_i)_- = f_{i,-},\ |g_i(K,V_i)_-| \le \left|\partial(g_i)_{(i)}\right|.$$

After replacing f_i by $f_i - f_{i,-}$ we can reduce to the case that $f_{i,-}=0$. Then it follows from the condition[23] that for $j>m$- we have $f_j\in k$-

$$|f_j| \le \left|\partial(f_i)_{(i)}\right|,$$

hence we are reduced to the case where $f_j=0$ for $j>m$. We then notice that for $U_i\cap U_j\ne\varnothing$ we have

$$\left|(f_i-f_j)|_{U_i\cap U_j}\right| = |(f_i-f_j)|_{D^{(i)}\cap D^{(j)}}|. \tag{**}$$

[21]Translator’s note: perhaps by k the author meant $\mathbb{A}^1$.

[22]Translator’s note: this we can see via exploring more on the decomposition mentioned above and the explicit maximal principle. One just need to be careful and check the inequality case by case.

[23]Translator’s note: the author perhaps was referring to the condition that $f_n=0$.

Now to every maximal element $D^{(\nu)}$ among $D^{(j)}$ with $j \leq m$ regarding "$\subset$" there is a $\mu > m$ with $U_\nu \cap U_\mu \neq \varnothing$, so that (**) implies that $|f_\nu| \leq \left|\partial(f_i)_{(i)}\right|$. Finally applying [5, Hilfssatz 2.4][24] to every connected component of

$$Z = \bigcup_{j=1}^{m} U_j$$

and notice (**) we get that

$$\max_{1\leq j\leq m} |f_j| \leq \left|\partial(f_i)_{(i)}\right|.$$

□

□

Remark 2. The proof of Satz 7 naturally generalizes to deduce that for each open cover $\mathcal{U} = \mathcal{U}(\underline{\omega}, \underline{\epsilon}, \underline{\epsilon}')$ with $\underline{\epsilon}' \leq 1$ of a reduced affinoid space $X \times E^1$ the group $H^1_\rho(\mathcal{U})$ vanishes.

3. For the convenience of the reader we formulate in advance

Hilfssatz 9. *Let X be a reduced affinoid space with an open cover $\mathcal{W} = (W_j)_{(j\in J)}$. If there is $b \in k^*$ with $|b| \leq 1$, so that to every inflated open cover $\mathcal{U}_\epsilon$ of X and every $f \in Z^1_\rho(\mathcal{U}_\epsilon)$ $\left(\rho \in \mathbb{R}^*_+\right)$, there exists an open cover $\mathcal{B}_j$ of W_j with*

$$\mathcal{B}_j \gg \mathcal{U} \cap W_j, \quad b \cdot f|_{\mathcal{B}_j} \in B^1_\rho(\mathcal{B}_j) \text{ for } j \in J.$$

Then there is $c \in k^*$ with $|c| \leq 1$ *and*

$$c \cdot H^1_\rho(X) = 0$$

(Here c does not depend on ρ, if this applies to b.[25])

Proof. According to Satz 6 it suffices to show that c can be chosen that for an open cover $\mathcal{B}$ of X such that $\mathcal{B} \gg \mathcal{U}_\epsilon$ we have $c \cdot f|_{\mathcal{B}} \in B^1_\rho(\mathcal{B})$. We reduce to the case where $\mathcal{B}_j$ is of the form $\mathcal{B} \cap W_j$ for some open cover $\mathcal{B} \gg \mathcal{U}_\epsilon$ of X. To that end, for $\varnothing \neq I \subset J$ let

$$W_I := \bigcap_{j\in I} W_j, \quad \mathcal{B}_I := \bigvee_{j\in I} (\mathcal{B}_j \cap W_I).$$

The family $\mathcal{B}$ whose elements consists of all of elements of $\mathcal{B}_I$ is an open cover of X with

$$\mathcal{B} \gg \mathcal{U}_\epsilon, \quad \mathcal{B} \cap W_j \gg \mathcal{B}_j \text{ for } j \in J.$$

Now let $(\mathcal{B}_j)_{(j)} = (\mathcal{B} \cap W_j)_{(j)}$. According to Banach's theorem

$$C^0(\mathcal{W}) \xrightarrow{\partial^0_{\mathcal{W}}} Z^1(\mathcal{W})$$

has a right inverse with norm $\leq |a^{-1}|$, where $a \in k^* \cap \mathring{k}$ is sufficiently small. We show that one can choose $c := ab$. We finish the proof in the following. Let $f \in Z^1_\rho(\mathcal{U}_\epsilon)$ with

$$b \cdot f|_{\mathcal{B}\cap W_j} = \partial^0(g_i^{(j)})_{(i)}$$

[24]Translator's note: even without the reference one should be able to finish the rest of the argument easily. In one sentence, we just keep trying with maximal principle and ultra-triangle inequality.

[25]Translator's note: perhaps the author meant that c can be defined in terms of b without involving ρ as long as the b satisfies the condition above. But as one can see the condition intrinsically involves ρ.

where $(g_i^{(j)})_{(i)} \in C^0_\rho(\mathcal{B} \cap W_j)$. Then by the way it is defined, we have

$$g^{(j,t)}|_{W_j \cap W_t \cap V_i} \coloneqq g_i^{(j)} - g_i^{(t)}$$

is an element[26] $(g^{(j,t)})_{(j,t)} \in Z^1_\rho(\mathcal{W})$ and hence

$$a(g^{(j,t)})_{(j,t)} = \partial^0_{\mathcal{W}}(g_j)_{(j)}$$

for some $(g_j)_{(j)} \in C^0_\rho(\mathcal{W})$. For $(h_i)_{(i)} \in C^0_\rho(\mathcal{B})$ with

$$h_i|_{W_j \cap V_i} \coloneqq a g_i^{(j)} - g_j{}^{27}$$

we have $ab \cdot f = \partial^0_{\mathcal{B}}(h_i)_{(i)}$. □

Proof of Theorem 2. Due to Hilfssatz 3 we may assume that the atlas satisfying conditions in Satz 4 is $= \{X\}$. Since the Zariski topology on X is Noetherian, it follows from Satz 4 that there are finitely many finite morphisms

$$\Phi_\sigma \colon X \to E^n, \quad \sigma = 1, \dots, s$$

as in Satz 4 with elements $a_\sigma \in \mathring{A}$, minimal polynomials Ω_σ and discriminants f_σ, so that

$$\bigcap_{\sigma=1}^{s} V(\phi_\sigma^*(f_\sigma)) = \varnothing.$$

Then there is $1 \geq \epsilon = |b| \in |k^*|$ sufficiently small, so that

$$\mathcal{W} = (W_\sigma)_{(\sigma)}, \quad W_\sigma \coloneqq \Phi_\sigma^{-1}(\{|f_\sigma| \geq \epsilon\}),$$

is an open cover of X. Then for any open cover $\mathcal{S}$ of E^n and any $q \geq 0$, $\sigma = 1, \dots, s$

$$\phi_\sigma^*(f_\sigma) \cdot C^q_\rho(\Phi_\sigma^{-1}(\mathcal{S})') \subset \sum_{0 \leq \nu < \deg \Omega_\sigma} \phi_\sigma^*(C^q_\rho(\mathcal{S})) \cdot a_\sigma^\nu,$$

c.f. [5, Page 57–58][28]. By Theorem 1 and Satz 3, for every $f \in Z^1_\rho(\mathcal{U}_\epsilon)$ where $\mathcal{U}_\epsilon$ is an inflated cover of X and for every σ there exists an open cover $\mathcal{S}^{(\sigma)}$ of E^n with $\Phi_\sigma^{-1}(\mathcal{S}^{(\sigma)})' \gg \mathcal{U}_\epsilon$ and

$$b \cdot f|_{\Phi_\sigma^{-1}(\mathcal{S}^{(\sigma)})' \cap W_\sigma} \in B^1_\rho(\Phi_\sigma^{-1}(\mathcal{S}^{(\sigma)})' \cap W_\sigma).$$

Let

$$\mathcal{B} \coloneqq \bigvee_{\sigma=1}^{s} \Phi_\sigma^{-1}(\mathcal{S}^{(\sigma)})',$$

then for all σ we have

$$b \cdot f|_{\mathcal{B} \cap W_\sigma} \in B^1_\rho(\mathcal{B} \cap W_\sigma).$$

The assumption of Hilfssatzes 3 are thus verified. □

[26]Translator's note: observe that the function defined above glue to a function over $W_j \cap W_t$
[27]Translator's note: observe again that these functions glue.
[28]Translator's note: once again, the translator has to work this out himself, see Lemma A.4

Proof of Theorem 2'. This proof is largely analogous as above. By arguing with irreducible components of X we can reduce to the case where $\widetilde{X}$[29] is irreducible. According to Satz 4', we get finitely many finite morphisms

$$\Phi_\sigma : X \to E^n, \quad \sigma = 1, \ldots, s;$$

with elements $\widetilde{a}_\sigma \in \widetilde{A}$, minimal polynomials $\widetilde{\Omega}_\sigma$ and $\widetilde{f}_\sigma$ as in Satz 4', so that

$$\bigcap_{\sigma=1}^{s} V(\widetilde{\phi}_\sigma^*(\widetilde{f}_\sigma)) = \varnothing.$$

By a main result of [9], the number of fibres of Φ_σ and $\widetilde{\Phi}_\sigma$ are the same. With the notations in the proof of Theorem 2 we can choose $b = 1$; it follows

$$f|_{\mathcal{B}\cap W_\sigma} \in B^1_\rho(\mathcal{B} \cap W_\sigma).$$

By Satz 5 $H^1_\rho(\mathcal{W}) = 0$, hence we have $f|_{\mathcal{B}} \in B^1_\rho(\mathcal{B})$.[30] □

Appendix A. Translator's final note

The translator thinks it would be helpful to record some auxiliary lemmata here.

Lemma A.1. *Let $\mathcal{U}$ and $\mathcal{B}$ be two open covers of a site X, let $\mathcal{F}$ be a sheaf of abelian groups on X and let $[f] \in \check{H}^1(\mathcal{B}, \mathcal{F})$. Suppose that*

(1) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and;

(2) for each $U \in \mathcal{U}$, we have $[f]|_{\mathcal{B}\cap U} = 0$.

Then we have

$$[f] = [0]$$

in $\check{H}^1(\mathcal{B}, \mathcal{F})$.

Proof. One may consider the double complex given by refinement of $\mathcal{U}$ and $\mathcal{B}$ and do a chase diagram argument.

More concretely (which might be more confusing), let us fix some notations. Suppose $[f]$ is represented by $f_{j_0j_1} \in \mathcal{F}(B_{i_0} \cap B_{i_1})$. Then by condition (2), there exist $f^i_j \in \mathcal{F}(U_i \cap B_j)$ such that

$$f_{j_0j_1} = f^i_{j_0} - f^i_{j_1} \quad \text{over} \quad U_i \cap B_{j_0} \cap B_{j_1}.$$

Then we see that $g^{i_0i_1}_j := f^{i_0}_j - f^{i_1}_j$ glue to $g^{i_0i_1} \in \mathcal{F}_{U_{i_0}\cap U_{i_1}}$ which is easily checked to be a cocycle in $\check{H}^1(\mathcal{U}, \mathcal{F})$. Therefore by condition (1) we can get $g^i \in \mathcal{F}(U_i)$ such that

$$g^{i_0i_1} = g^{i_0} - g^{i_1} \quad \text{over} \quad U_{i_0} \cap U_{i_1}.$$

Finally let $\widetilde{f^i_j} := f^i_j - g^i \in \mathcal{F}(U_i \cap B_j)$, we just observe that these glue to give sections $f_j \in \mathcal{F}(B_j)$ whose Čech differential is exactly $f_{j_0j_1} \in \mathcal{F}(B_{j_0} \cap B_{j_1})$. □

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[29]Translator's note: it seems that by $\widetilde{X}$ the author meant the smooth reduction of X. However, the translator thinks the argument will be neater if one think of $\widetilde{X}$ as an affine smooth formal model of X. Therefore, from now on, we treat $\widetilde{X}$ as an affine smooth formal model of X.

[30]Translator's note: the author has implicitly used auxiliary Lemma A.1 here.

Lemma A.2. *Let $n = d + 1$. Let $\omega \in \mathring{T}_d\langle Y\rangle$ be a norm 1 Weierstrass polynomial of degree $s > 0$. Then every element $f \in T_n\langle S\rangle/(1 - \omega \cdot S)$ admits a unique decomposition*

$$f = h + \sum_{\nu>0} r_\nu S^\nu \qquad \mathrm{mod}\ (1 - \omega \cdot S),$$

where $h \in T_n$ and all the $r_\nu \in T_d[Y]$ are of degree $< s$. Moreover we have the following inequality between the norms:

$$|h| \leq |f|.$$

Proof. This follows from Weierstrass division, c.f. Theorem 8 in Bosch's Lectures on Formal and Rigid Geometry. Details to be added. ☐

Lemma A.3. *In the situation of Satz 7, we can analytically continue f_{2^m} over U'.*

Proof. Over $U'_{|\omega|\geq 1}$, we see that $f_{2^m} = \sum_{\nu<0} r_\nu \omega^\nu$ naturally converges to an analytic function. We also observe that, since $\omega \in \mathring{K}[Y]$ is a norm 1 Weierstrass polynomial, $|\omega| \leq 1 \iff |Y| \leq 1$. Therefore we see that $U'_{|\omega|\leq 1} = U'_{|Y|\leq 1}$, and f_{2^m} is already given there. On the overlap, namely $U'_{|\omega|=1}$, these two functions coincide by our condition. ☐

Lemma A.4. *Let A be an irreducible n-dimensional affinoid algebra over a non-archimedean field K. Assume there is a finite morphism $T_n \to A$ and an element $a \in A^\circ$ with minimal polynomial $\Omega \in T_n^\circ[X]$ of degree d whose discriminant is $f \in T_n^\circ$ such that*

$$T_n[x]/(\Omega)[f^{-1}] = A[f^{-1}]. \tag{***}$$

Then for every affinoid subdomain $\mathrm{Sp}(B) \subset \mathrm{Sp}(T_n)$ we have that

$$f \cdot (A\hat{\otimes}_{T_n} B)^\circ \subset \sum_{0\leq\nu<d} B^\circ \cdot a^\nu.$$

Proof. Step 1: we first prove this when $n = 0$. In this case, due to the fact that $\mathcal{O}_{\bar{K}} \cap K = \mathcal{O}_K$, we can assume that K is algebraically closed. Let us reset the situation. So let $L = \prod_{i=1}^d K_i$ be a finite K-algebra where all of K_i's are isomorphic to K. Let $y = (y_i) \in \mathcal{O}_L = \prod_{i=1}^d \mathcal{O}_{K_i}$. Then the minimal polynomial Ω of y over K is given by

$$\Omega = \prod_{i=1}^d (x - y_i).$$

To make the problem less trivial, let us assume that the discriminant of Ω:

$$f = \prod_{i\neq j} (y_i - y_j) \neq 0.$$

So the condition *** translated to that $L = K[x]/(\Omega)$. Finally let $\alpha = (\alpha_i) \in \mathcal{O}_L = \prod_{i=1}^d \mathcal{O}_{K_i}$, then we know that $\alpha \cdot f = \sum_{0\leq\nu<d} \beta_\nu \cdot y^\nu$ for some $\beta_i \in K$. We need to show that in this case, $\beta_i \in \mathcal{O}_K$. To that end, we simply observe that we have a set of equations

$$M := \begin{bmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{d-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_d & y_d^2 & \cdots & y_d^{d-1} \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{bmatrix} = \begin{bmatrix} \alpha_1 \cdot f \\ \alpha_2 \cdot f \\ \vdots \\ \alpha_d \cdot f \end{bmatrix}$$

Since $f = (\det M)^2$ and M, M^{ad} has coefficients in $\mathcal{O}_K$ where M^{ad} is the adjugate matrix of M, we see that β_i's are in $\mathcal{O}_K$.

Step 2: now we have to use Riemannsche Hebbarkeitssatz. Let $\alpha \in (A\hat{\otimes}_{T_n}B)^\circ$, then due to condition *** we see that

$$f \cdot \alpha = \sum_{0\le\nu<d} \beta_i \cdot a^\nu \quad \text{over } \operatorname{Sp}(A\hat{\otimes}_{T_n}B)\backslash V(f)$$

for some $\beta_i \in B[f^{-1}]$. By previous case we know that β_i remains norm ≤ 1 on $\operatorname{Sp}(B)\backslash V(f)$. By Riemannsche Hebbarkeitssatz we know that $\beta_i \in B^\circ$ which is what we want to show. □

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References

1. Wolfgang Bartenwerfer, *Der Kontinuitätssatz für reindimensionale k-affinoide Räume*, Math. Ann. **193** (1971), 139–170. MR 0287036
2. ______, *Die nichtarchimedische Version des Rothsteinschen Kontinuitätssatzes*, Math. Ann. **202** (1973), 89–134. MR 0382707
3. ______, *Die Fortsetzung holomorpher und meromorpher Funktionen in eine k-holomorphe Hyperfläche hinein*, Math. Ann. **212** (1975), 331–358. MR 0402114
4. ______, *Der erste Riemannsche Hebbarkeitssatz im nichtarchimedischen Fall*, J. Reine Angew. Math. **286/287** (1976), 144–163. MR 0422680
5. ______, *Ein Konvergenzsatz für* 0*-Koketten affinoider Funktionen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1976), no. 4, 1–30. MR 0466613
6. Dieter Denneberg, *Divisoren in eindimensionalen affnoiden und affinen Räumen*, Math. Ann. **181** (1969), 137–151. MR 0248133
7. Lothar Gerritzen, *über die picard-gruppe nichtarehimedischer holomorpher räume*, Unpublished Manuscript.
8. Lothar Gerritzen and Hans Grauert, *Die Azyklizität der affinoiden überdeckungen*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 159–184. MR 0255854
9. Hans Grauert and Reinhold Remmert, *über die Methode der diskret bewerteten Ringe in der nichtarchimedischen Analysis*, Invent. Math. **2** (1966), 87–133. MR 0206039
10. Reinhardt Kiehl, *Die de Rham Kohomologie algebraischer Mannigfaltigkeiten über einem bewerteten Körper*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 33, 5–20. MR 0229644
11. ______, *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math. **2** (1967), 256–273. MR 0210949
12. John Tate, *Rigid analytic spaces*, Invent. Math. **12** (1971), 257–289. MR 0306196

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