(VERY BRIEF) LECTURES ON FORMAL-RIGID GEOMETRY

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1. An introduction to Rigid Geometry

1.1. Non-Archimedean fields.

A *non-archimedean field* is a field which is complete under a *non-archimedean absolute value*, i.e., an absolute value $\|\cdot\|$ satisfying

(1) $\|x\| = 0 \iff x = 0$;
(2) $\|xy\| = \|x\| \|y\|$ and;
(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$.

There are three main examples of these fields, namely $\mathbb{F}_p(\!(T)\!)$, $\mathbb{C}(\!(T)\!)$ and $\mathbb{Q}_p$.

1.2. Motivating example of Tate.

Giving an elliptic curve over $\mathbb{C}$, one may view it as a quotient of $\mathbb{C}^*$ in the category of complex manifolds. Can we do something like this over non-archimedean field? Tate observed the following example.

Example 1.1. Let K be an algebraically closed non-archimedean field, for example we may take $K = \mathbb{C}_p$. Using T as a variable, look at the algebra

$$\mathcal{O}(K^*) = \left\{ \sum_{i \in \mathbb{Z}} a_i T^i \;\middle|\; a_i \in K, \lim_{|i| \to \infty} \|a_i\| r^i = 0 \text{ for all } r > 0 \right\}.$$

Now let $a \in K$ be an element with $0 < \|a\| < 1$, and write

$$\mathcal{M}^a(K^*) := \{ f \in \mathcal{O}(K^*) \mid f(aT) = f(T) \}.$$

Tate made the observation that $\mathcal{M}^a(K^*)$ is the function field of an elliptic curve (over K) with a non-integral j-invariant, i.e., with $\|j\| > 1$. Furthermore, he saw that the set of K-points of aforementioned elliptic curve coincides canonically with the quotient $K^*/a^{\mathbb{Z}}$.

Elliptic curves obtained in this fashion are called *Tate curves*. In fact, an elliptic curve is a Tate curve if and only if it has non-integral j-invariant in the above sense. Tate proposed the subject later known as *Rigid Geometry* so that one is allowed to consider objects like $K^*/a^{\mathbb{Z}}$ (notice that this quotient is meaningless in the setting of algebraic geometry).

1.3. Tate algebras.

From now on, let us fix a non-archimedean field K.

Let us ask ourselves the following question: what should be "the ring of functions" on a closed disc[1] of radius 1 over K? In complex geometry any holomorphic function

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[1]Notice that closed discs are open in the world of non-archimedean geometry due to strong triangle inequality.

f on an open disc can be Taylor-expended around the origin:

$$f = \sum_{n \geq 0} a_n z^n,$$

the only condition is that this series should converge on the open disc. Therefore, it is natural to say that the sought-after ring of functions is the ring of *convergent* power series $\sum_{n \geq 0} a_n T^n$ where T is the coordinate function on our closed disc. One can check that the convergent condition in our setting translates into

$$\lim_{n \to \infty} \|a_n\| = 0.$$

Similarly we can consider convergent power series $\sum_I a_I T^I$ on polydisc, here I denotes multi-indices, the convergent condition translates into

$$\lim_{|I| \to \infty} \|a_I\| = 0.$$

After discussion above, let us make the following definition.

Definition 1.2. The Tate algebra of n variables over K is defined to be

$$K\langle T_1, \ldots, T_n \rangle := \left\{ \sum_I a_I T^I \,\middle|\, \lim_{|I| \to \infty} \|a_I\| = 0 \right\}.$$

We often denote $K\langle T_1, \ldots, T_n \rangle$ by T_n.

These rings carry a natural structure of Banach algebras. Indeed, we define the norm by

$$\left\| \sum_I a_I T^I \right\| := \max \{\|a_I\|\}. \qquad (\boxdot)$$

Given any $f \in K\langle T_1, \ldots, T_n \rangle$ and any K-point $\underline{x}$ in the closed polydisc of radius 1, we may evaluate $f(\underline{x}) \in K$. In particular, we can consider $\|\cdot(\underline{x})\|$ for any K-point $\underline{x}$. These define a family of semi-norms on $K\langle T_1, \ldots, T_n \rangle$. One can check that the supremum of these semi-norms is the norm defined in $\boxdot$.

Tate algebras share a lot of good properties as polynomial algebras, let us summarize some of them in the following.

Proposition 1.3.

- *The ring T_n is*
  1. *Noetherian;*
  2. *of Krull dimension n;*
  3. *regular;*
  4. *Jacobson;*
  5. *Nagata and;*
  6. *Excellent.*
- *The residue field of any maximal ideal $\mathfrak{m} \subset T_n$ is finite over K.*
- *The natural map*

$$\mathbb{D}^n(\overline{K}) \to Max\ T_n,\ x \mapsto \mathfrak{m}_x := \{f \in T_n \mid f(x) = 0\}$$

is surjective. Here $\mathbb{D}^n$ denotes the closed polydisc of dimension n and radius 1.

The key techniques of proving the above statements are Weierstraß Preparation and Division (similar as in the theory of multi complex variables). We refer the interested reader to [Bos14, Section 2.2, Theorem 8, Corollary 9].

1.4. **Affinoid algebras.** In general, we should consider "rigid affine" objects, namely the zero locus of finitely many analytic functions inside polydiscs. Therefore we make the following definition.

Definition 1.4. A K-algebra A is called an *affinoid K-algebra* if there is an surjection of K-algebras $\alpha\colon T_n \twoheadrightarrow A$.

Remark 1.5.

(1) Notice that since Tate algebras are Noetherian, the kernel of a chosen surjection α is always a finitely generated ideal. Being quotient of a Noetherian algebra, we see that affinoid algebras are also Noetherian.
(2) One can show that finitely generated ideals in T_n are always closed with respect to the norm defined in ⊡. In particular, any presentation defines a norm on the affinoid algebra A. Any two different presentations define *equivalent* norms. We call these norms by *residue norms.*
(3) On the other hand, the evaluation on closed points (i.e., maximal ideals) still defines a family of semi-norms. The supremum of these semi-norms only defines a semi-norm in general. If A is reduced, then this semi-norm is equivalent to the residue norms.

The most important fact concerning affinoid algebras is the analogue of Noether's normalization.

Proposition 1.6 (Noether's normalization)**.** *Let A be an affinoid K-algebra, there exists $d \in \mathbb{N}$ and a finite injective algebra homomorphism*

$$T_d \hookrightarrow A.$$

Moreover, d is uniquely determined as the Krull dimension of A.

In the theory of schemes, one associates any ring with a triple $(X, \tau, \mathcal{O})$ consisting of a set X (of prime ideals in A), a topology τ on X (i.e., the Zariski topology) and a structure sheaf $\mathcal{O}$. In our setting of rigid geometry, first thing we associates any affinoid K-algebra with is the set X of *maximal ideals* inside A, which people sometimes denote as $\mathrm{Max}(A)$. The reason for only considering maximal ideals is quite subtle, we refer the interested reader to [Bos14, Page 60-63]. The second thing we need is a "topology"[2] on $\mathrm{Max}(A)$. It is tempting to just define such a topology using "distance" coming from the embedding of $\mathrm{Max}(A)$ inside some polydisc. A few seconds' thought says this is a terrible idea. Take the closed disc $\mathbb{D}^1$ for example, being equipped with the topology coming from non-archimedean distance makes it totally disconnected! There would be too many functions on it. A fix of this is given by Tate: "In 1961 Tate introduced such an extra structure and overcame the disconnectedness of the topology of a non-Archimedean field. Thus, he saved the analytic continuation and the identity principle over totally disconnected ground fields, hence making the impossible possible, as Remmert said." (A quote from [Lüt16, Introduction].)

We will have a glance at this extra structure in the next section.

[2] I'm whispering Grothendieck before saying topology.

1.5. **Rational subdomains/coverings.**

Definition 1.7. Let A be an affinoid K-algebra and $f_1, \ldots, f_n, g \in A$ elements which generate unit ideal in A. Consider the subset of $X \coloneqq \mathrm{Max}(A)$ given by

$$\{x \in \mathrm{Max}(A) \mid \|f_i(x)\| \leq \|g(x)\|\}.$$

We denote it by $X\langle \frac{f_i}{g} \rangle$. Subsets of this type are called *rational subdomains.*

Remark 1.8. The condition of $(f_1, \ldots, f_n, g) = (1)$ implies that for any $x \in X$, the norms $\|f_i(x)\|, \|g(x)\|$ are not all zero.

In the situation above, we may consider another affinoid algebra

$$A\langle \frac{f_i}{g} \rangle \coloneqq A\langle T_i \rangle / (g \cdot T_i - f_i).$$

As suggested by the notation, we have the following lemma.

Lemma 1.9. *The natural map $A \to A\langle \frac{f_i}{g} \rangle$ induces a canonical bijection*

$$\mathrm{Max}(A\langle \frac{f_i}{g} \rangle) \xrightarrow{\cong} X\langle \frac{f_i}{g} \rangle.$$

Moreover, the affinoid K-algebra $A\langle \frac{f_i}{g} \rangle$ only depends on the rational subdomain $X\langle \frac{f_i}{g} \rangle$.

One imagine that $\mathrm{Max}(A\langle \frac{f_i}{g} \rangle)$ is an "affine open" inside $\mathrm{Max}(A)$. The following lemma partly confirms that.

Lemma 1.10. *$A\langle \frac{f_i}{g} \rangle$ is a flat A algebra. They have the same completed local ring at points $x \in X\langle \frac{f_i}{g} \rangle$.*

The rational subdomains are the basic "opens" of the (Grothendieck) topology that we alluded to in previous subsection. After introducing them, let us define basic "coverings" of our (Grothendieck) topology.

Definition 1.11. Let $f_0, f_1, \ldots, f_n \in A$ be elements in A which generate the unit ideal in A. It is easy to see that the subsets $X_i \coloneqq X\langle \frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i} \rangle$ form a covering

$$X = \bigcup_i X_i$$

of $X \coloneqq \mathrm{Max}(A)$. Coverings of this type are called *rational coverings.*

Rational subdomains and rational coverings put a weak Grothendieck on an affinoid. To this point, one has to wave his/her hands and refer interested readers to literature for the precise definition of Grothendieck topology on the set of maximal ideals in an affinoid algebra, c.f. [Bos14, Section 5.1]. In principle, it does not hurt to just consider rational subdomains and rational coverings. In the following we will only consider these to be the "admissible" opens and coverings, the readers should be cautious about this issue[3]. After introducing this correction of naïve topology, we should define a structure sheaf with respect to this topology. This is carried out in the next subsection.

[3] The topos corresponding to this coarser site is actually the same as the Grothendieck topos, assuming one only considers affinoids.

1.6. **Coherent sheaves, Theorem A and B à la Cartan/Tate.** Throughout this subsection, let A be an affinoid K-algebra and let $X := \mathrm{Max}(A)$. Let M be a finitely generated A module, e.g., $M = A$. We may consider the following presheaf:

$$\widetilde{M}(X\langle\frac{f_i}{g}\rangle) := M \otimes_A (A\langle\frac{f_i}{g}\rangle).$$

Given $f_0, f_1, \ldots, f_n$ in A generating the unit ideal, one can form rational covering $X = \bigcup_i X_i$. Therefore one can consider the Čech complex associated with $\widetilde{M}$ and this covering $\mathcal{U} = \{X_i\}$, which we denoted as $\check{C}^\bullet(\widetilde{M}, \mathcal{U})$. The following theorem of Tate is a reincarnation of the fundamental fact in algebraic/complex geometry that coherent sheaves are acyclic on affine/stein spaces (Theorem A and B à la Cartan).

Theorem 1.12 (Tate). *$\widetilde{M}$ is a sheaf, with no higher cohomology on X. More precisely, for any rational covering $\mathcal{U}$ we have*

$$M = \check{H}^0(\widetilde{M}, \mathcal{U}),$$

and

$$\check{H}^i(\widetilde{M}, \mathcal{U}) = 0$$

for all $i > 0$.

Definition 1.13. Sheaves of the form $\widetilde{M}$ for some finitely generated A module M are called *coherent sheaves*. Taking $M = A$, one gets the *structure sheaf* $\mathcal{O}_X = \widetilde{A}$.

So by now, we know what an *affinoid rigid space* is. It is just a triple consisting of a set, an underlying (Grothendieck) topology and a structure sheaf associated with an affinoid K-algebra. One denotes the affinoid rigid space associated with an affinoid K-algebra A by $\mathrm{Sp}(A)$. Finally we are prepared to make the following definition.

Definition 1.14. A *rigid space over K* is a locally ringed (Grothendieck) topological space[4] which is locally isomorphism to an affinoid rigid space associated with an affinoid K-algebra.

1.7. **Properness and finiteness.** In complex geometry, one formulation of properness (of a complex manifold M) is to exhibit two finite coverings of polydiscs $(\mathcal{U}, \mathcal{V})$ such that for each U_i there is a V_i with $U_i \subset\subset V_i$. The latter just means the closure of U_i (inside M) is contained in V_i. One uses functional analysis, in particular the theory of compact operators, to show finiteness of cohomology of coherent sheaves on proper complex manifolds. The same story is carried out by Kiehl.

Definition 1.15. Let $\mathrm{Sp}(A) \subset \mathrm{Sp}(B)$ be two affinoid spaces. We say $\mathrm{Sp}(A)$ is strictly contained in $\mathrm{Sp}(B)$ if there is a presentation $T_n(\underline{Z}) \twoheadrightarrow B$ of B such that

$$\mathrm{Sp}(A) \subset \mathrm{Sp}(B)\langle\frac{Z_i}{a}\rangle$$

for some $a \in K$ with $\|a\| < 1$.

A rigid space X over K is proper if there are two (admissible) finite coverings of affinoid spaces $\mathcal{U} = \{\mathrm{Sp}(A_i)\}$ and $\mathcal{V} = \{\mathrm{Sp}(B_i)\}$ such that each $\mathrm{Sp}(A_i)$ is strictly contained in $\mathrm{Sp}(B_i)$ for all i.

[4]One can be fancy and say locally ringed site

In the next section, we will see an (easier) interpretation of properness in terms of properness in algebraic geometry. Following the same strategy as complex analytic case, Kiehl proves the finiteness of cohomology.

Theorem 1.16 (Kiehl)**.** *Let $\mathcal{F}$ be a coherent sheaf on a proper rigid space X. Then $H^i(X, \mathcal{F})$ is a finite dimensional K-vector space.*

The above theorem also has a version for proper morphisms. One can actually use the above theorem to show the coherence of proper push-forward of coherent sheaves in algebraic geometry. The key to prove the above theorem is the following lemma.

Lemma 1.17. *If $\mathrm{Sp}(A)$ is strictly contained in $\mathrm{Sp}(B)$, then the natural morphism $B \to A$ is a compact operator.*

Here we put residue norms on A and B so that they are (non-archimedean) Banach algebras, hence compact operator makes sense. The point is that under the setting of non-archimedean Banach spaces, the functional analysis is actually easier ....

Let us work out the example of $\mathrm{Sp}(T_1(\frac{Z}{p})) \subset \mathrm{Sp}(T_1(Z))$ where $K = \mathbb{Q}_p$.

1.8. **Examples.** The examples that I plan to cover are: Tate curves, any algebraic varieties, GAGA, Abeloid varieties, Hopf surfaces.

2. Formal models à la Raynaud

There are four approaches to non-archimedean geometry: Tate's approach (roughly sketched in previous section), Raynaud's approach using formal geometry, Berkovich's analytic spaces and Huber's adic spaces. These four have different features, we plan to roughly introduce Raynaud's approach in this section. Interested readers are highly recommended to read the foundational series [BL93a], [BL93b], [BLR95a] and [BLR95b].

2.1. **generic fibre of a formal scheme.** Throughout this section, let $\mathcal{O}$ be a complete discrete valuation ring with $\pi \in \mathcal{O}$ its uniformizer. Let $K = \mathrm{Frac}(\mathcal{O})$. Given a scheme over $\mathrm{Spec}(\mathcal{O})$, one can talk about its generic fibre as a scheme over $\mathrm{Spec}(K)$. We naturally wonder if one could make sense of the "generic fibre" of a formal scheme over $\mathrm{Spf}(\mathcal{O})$. Raynaud's theory provides such a perspective to view the generic fibre as a rigid space. Let us take a look at affine situations.

Example 2.1. Consider $T_n^\circ := \mathcal{O}\langle Z_1, \ldots, Z_n\rangle$, which is the sub-algebra inside T_n consisting of elements with norm ≤ 1. This is a Noetherian adic ring with ideal $\pi \cdot T_n^\circ$ defining the topology. In particular, $\mathrm{Spf}(T_n^\circ)$ is an affine formal scheme over $\mathrm{Spf}(\mathcal{O})$. It is natural to assign the affinoid rigid space $\mathrm{Sp}(T_n)$ to be the generic fibre. For some reason, we will call it the rigid generic fibre and denoted as $\mathrm{Spf}(T_n^\circ)^{\mathrm{rig}}$.

Slightly more generally, consider an ideal I in T_n° which satisfies the condition

$$\{a \in T_n^\circ \,|\, \pi^m \cdot a \in I \text{ for some m}\} = I.$$

Then $\mathcal{A} := T_n^\circ/I$ is an adic flat $\mathcal{O}$-algebra with $\pi \cdot \mathcal{A}$ defining its topology. Similarly we can take $\mathrm{Sp}(A)$, where $A := \mathcal{A}[1/\pi]$, to be $\mathrm{Sp}(\mathcal{A})^{\mathrm{rig}}$.

Definition 2.2. An $\mathcal{O}$-algebra is *admissible* if it is a quotient of T_n° and flat over $\mathcal{O}$ (i.e., no π-torsion). A formal scheme $\mathcal{X}$ over $\mathcal{O}$ is *admissible* if it is covered by formal spectrum of admissible $\mathcal{O}$-algebras.

Example 2.3. Let A be an affinoid K-algebra, consider A° be the sub-algebra of power bounded elements in A. Assume for the moment that A° is an admissible $\mathcal{O}$-algebra. Then one can define a specialization map

$$\mathrm{Max}(A) \to \mathrm{Max}(A_{\mathrm{red}} \coloneqq \frac{A^\circ}{\pi \cdot A^\circ}) \text{ by } \mathfrak{m} \mapsto (\mathfrak{m} \cap A^\circ + \pi \cdot A^\circ).$$

One can check that this specialization map is surjective.

Lemma 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two admissible $\mathcal{O}$-algebras. If $\mathcal{B} \to \mathcal{A}$ induces an open embedding of* $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$*, then it induces an open embedding of* $\mathrm{Sp}(A[1/\pi]) \to \mathrm{Sp}(B[1/\pi])$.

Example 2.5. In the case when $\mathcal{B} = \mathcal{A}\langle 1/f \rangle$, which corresponds to a fundamental open. The associated open subspace is just $\mathrm{Sp}(A)\langle 1/f \rangle$.

Proposition 2.6. *By the discussion above, one can assign a rigid space to any admissible* quasi-paracompact *formal scheme over $\mathcal{O}$ as its generic fibre.*

The construction is very standard. In one sentence we take an arbitrary locally finite covering by affine admissible formal schemes, take their generic fibre and glue the overlaps. This gives a functor of assigning rigid generic fibres:

$$(\boxdot) \qquad \left\{ \begin{matrix} \text{category of admissible quasi-paracompact} \\ \text{formal schemes over } \mathcal{O} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of rigid} \\ \text{spaces over } K. \end{matrix} \right\}$$

It is natural to ask whether this map is injective, or if it is (essentially) surjective. These two questions are answered by Raynaud's theorem.

2.2. **admissible blow-up.** In this subsection we address the "fibre" of $\boxdot$.

Heuristic time: if we are literally taking generic fibre of some $\mathcal{O}$-scheme, then after modifying along the special fibre we would get the same outcome. In other words, an arbitrary blow-up with center supported on special fibre does not change the generic fibre.

Given an admissible $\mathcal{O}$-algebra $\mathcal{A}$, and a finitely generated ideal I. One can form a *formal blow-up* of the formal scheme $\mathrm{Spf}(\mathcal{A})$ by usual blow-up and then perform π-adic completion.[5] We have special interest in the case where $I \supset (\pi^r)$ for some r, equivalently I is an open ideal. More generally, given an admissible quasi-paracompact formal scheme $\mathcal{X}$ over $\mathcal{O}$ and a coherent sheaf $\mathcal{I}$ of open ideals. We can similarly form the *admissible formal blow-up* $\mathcal{X}_{\mathcal{I}}$, c.f. [Bos14, 8.2, Definition 3].

Example 2.7. Let $\mathcal{A} = \mathcal{O}\langle Z \rangle$, and let $I = (\pi, Z)$. The resulting formal blow-up has two affine charts. The generic fibre is still $\mathrm{Sp}(T_1)$, the generic fibre of two affine charts form a rational covering given by (π, Z).

This example illustrates the fact that performing admissible formal blow-up does not change the rigid generic fibre. Slightly more surprisingly, the converse also holds.

Proposition 2.8. *Let $\mathcal{X}$ be an admissible quasi-paracompact formal scheme, let $\mathcal{I}$ be a coherent sheaf of open ideals. Then there is a natural isomorphism*

$$\mathcal{X}_{\mathcal{I}}^{\mathrm{rig}} \to \mathcal{X}^{\mathrm{rig}}$$

of rigid spaces.

[5]Equivalently one can use the Proj construction, c.f. [Bos14, 8.2, Definition 3].

Conversely, if two admissible quasi-paracompact formal schemes $\mathcal{X}$ and $\mathcal{X}'$ have isomorphic rigid generic fibre. Then there exists a common admissible formal blow-up of both $\mathcal{X}$ and $\mathcal{X}'$ and the induced isomorphism of the rigid generic fibres agrees with aforementioned isomorphism.

The proposition above is not hard. Namely if we were locked in a room with only pen and paper and were told that we have to prove this proposition in order to get out, then we are highly likely to survive.

2.3. **Raynaud's Theorem.** One can use several equivalent ways to define the notion of *(quasi)-separatedness* of a rigid space. It is easy to see that the quasi-paracompactness of an admissible formal scheme implies the rigid generic fibre is quasi-separated and quasi-paracompact. Now we are prepared to state the following theorem of Raynaud.

Theorem 2.9 (Raynaud)**.** *The functor $\boxdot$ induces an equivalence between*

$$\left\{\begin{array}{c}\textit{category of admissible quasi-paracompact}\\ \textit{formal schemes over } \mathcal{O} \textit{ localized by}\\ \textit{the full subcategory of admissible formal blow-ups}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{category of quasi-separated}\\ \textit{quasi-paracompact rigid}\\ \textit{spaces over } K.\end{array}\right\}$$

Remark 2.10. (1) The theory of formal models provide a way to understand Grothendieck topology on a rigid space. Fix a quasi-separated quasi-paracompact (shorthand as qcqs from now on) rigid space X over K. Then by Raynaud's theorem, we can find a formal model $\mathcal{X}$. Consider

$$\widehat{\mathcal{X}} := \varprojlim_{\substack{\text{admissible}\\ \text{formal blow-}\\ \text{up } \mathcal{X}' \to \mathcal{X}}} |\mathcal{X}'|$$

as a topological space. Then the natural morphism (given by limit of specialization maps) $X \to \widehat{\mathcal{X}}$ is an isomorphism of sites.

(2) Let $\mathcal{X}$ be a formal model of a qcqs rigid space X, and let F be a coherent sheaf on X. Then one can always find a formal model $\mathcal{F}$ of F on $\mathcal{X}$.

2.4. **Examples, Néron model.** Depending time, I plan to talk about examples of formal models of $\mathbb{P}^1$, $\mathbb{G}_m$, Tate curves, Abeloid varieties, Hopf surfaces, etc.

One can define *smoothness* for formal schemes and rigid spaces. We can mimic the definition of Néron model to make the following.

Definition 2.11. Let X be a smooth rigid space, a *formal Néron model* of X is a pair consisting of a smooth formal scheme $\mathcal{X}$ and a morphism $f\colon \mathcal{X}^{\mathrm{rig}} \to X$ satisfying the obvious analogue of Néron mapping property.

If time permitted, I will address the relation between formal Néron models and Néron models and explain (a little bit) how it helps one to find a Raynaud's presentation of an abelian variety over a non-archimedean field.

2.5. **Flattening Technique.** Given a morphism $f\colon X \to Y$ of quasi-compact quasi-separated rigid spaces, we can find a formal model $\mathfrak{f}\colon \mathcal{X} \to \mathcal{Y}$ by Raynaud's theorem. One naturally asks if f has some good property, e.g. flatness, then can we find a good formal model of f?

Theorem 2.12 ([BL93b]). *Let $X \to Y$ be a morphism of qcqs rigid spaces, let F be a coherent sheaf on X which is Y-flat. Then one can find a formal model $\mathcal{F}$ on $\mathcal{X}$ and $\mathcal{X} \to \mathcal{Y}$ such that $\mathcal{F}$ is $\mathcal{Y}$-flat.*

The strategy to prove this theorem is to first pick an arbitrary formal model. Then using (formal version of) Gruson–Raynaud to perform admissible formal blow-up of the base so that the strict transformation would do the job.

Another worth-mentioning theorem along this line is the reduced fibre theorem of Bosch–Lütkebohmert–Raynaud.

Theorem 2.13 ([BLR95b]). *Let $X \to Y$ be a flat morphism of qcqs rigid spaces with geometrically reduced fibres. Then there is an étale surjection $Y' \to Y$ such that the base change to Y' has a formal model $\mathcal{X}' \to \mathcal{Y}'$ which is flat and has geometrically reduced fibres.*

2.6. **Other properties of formal models.** Some properties of (a morphism of) rigid spaces are completely captured by their formal models. Let us mention the following theorem due to Lütkebohmert–Huber–Temkin.[6]

Theorem 2.14. *Let X be a qcqs rigid space over K and let $\mathcal{X}$ be a formal model. TFAE*

(1) *X is separated (resp. proper);*
(2) *$\mathcal{X}_0$ is separated (resp. proper).*

Here $\mathcal{X}_0$ denotes the special fibre of $\mathcal{X}$.

This theorem implies the following proof of Kiehl's theorem of coherence of proper pushforward by using formal GAGA and coherence of proper pushforward in usual algebraic geometry.

3. Rigid spaces with projective reductions

This section is about my previous works. Let me just (shamelessly?) copy part of introductions of my papers here. . . .

In the famous series [BL93a], [BL93b], [BLR95a] and [BLR95b], Bosch, Lütkebohmert and Raynaud laid down the foundations relating formal and rigid geometry. The type of questions they treat in the series are mostly concerned with going from the rigid side to formal side. In this paper we will consider the opposite type of question, namely we will investigate to what extent properties on the formal side inform us about rigid geometry. More precisely, we will see what geometric consequences one can deduce under the assumption that the rigid space has a projective reduction.

Let K be a non-archimedean field with residue field k. Let X over K be a connected smooth proper rigid space with a K-rational point $x : \mathrm{Sp}(K) \to X$.

In this paper we prove the following:

Theorem 3.1. *Suppose that X has a formal model $\mathcal{X}$ whose special fiber $\mathcal{X}_0$ is projective over $\mathrm{Spec}(k)$, assume furthermore that Picard functor is represented by a quasi-separated rigid space. Then its identity component Pic^0_X is proper.*

Let us make a historical remark on the representability of Picard functor in rigid geometry.

[6] In our setting of discrete valuation, this theorem is proved by Lütkebohmert–Huber.

Remark 3.2.

(1) When K is discretely valued, Hartl and Lütkebohmert proved the representability of the Picard functor on the category of smooth rigid spaces over K under an additional assumption that X has a strict semistable formal model. They prove a structure theorem for the Picard space. In particular, the it is quasi-separated.
(2) In Warner's thesis it is proved that assuming K has characteristic 0, the Picard functor defined on a suitable category of adic spaces over K is represented by a separated rigid space over $\mathrm{Spa}(K, \mathcal{O})$.
(3) In general, we expect the Picard functor of a proper smooth rigid space over K to be represented by a separated rigid space.

If X has a projective reduction then one can naturally define an open and closed sub-functor $\underline{\mathrm{Pic}}^P_{X/K}$ of $\underline{\mathrm{Pic}}_{X/K}$.

Theorem 3.3 (Main Theorem)**.** *Suppose that X has a formal model $\mathcal{X}$ whose special fiber $\mathcal{X}_0$ is projective over* $\mathrm{Spec}(k)$*. Then $\underline{\mathrm{Pic}}^P_{X/K}$ is a proper functor.*

By a theorem of Kedlaya–Liu, it suffices to prove $\underline{\mathrm{Pic}}^P_{X/K}$ is bounded. We use moduli of semistable sheaves to show this.

Combining the above theorem with Lütkebohmert's structure theorem for proper rigid groups and the p-adic comparison results of Scholze, we deduce the following result.

Theorem 3.4. *Let X be a smooth proper rigid space over a p-adic field K. Assume that X has a formal model $\mathcal{X}$ over* $\mathrm{Spf}(\mathcal{O}_K)$ *whose special fiber is projective. Then we have*

$$h^{1,0}(X) = h^{0,1}(X).$$

Remark 3.5. By work of Conrad–Gabber we may generalize this Theorem to the situation where K is an arbitrary non-archimedean field extension of $\mathbf{Q}_p$.

This result suggests that the condition of admitting a formal model with projective reduction could be a natural rigid analytic analogue of the Kähler condition. In particular, it is natural to ask if this condition implies Hodge symmetry in higher degrees:

Conjecture 3.6. *Let X be a smooth proper rigid space admitting a formal model with projective reduction. Is it true that $h^{i,j}(X) = h^{j,i}(X)$ for all i, j?*

I guess I will stop here.

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